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On the large values of the Riemann zeta-function on short segments of the critical line¹⁾

Abstract. In this paper, we obtain a series of new conditional lower bounds for the modulus and the argument of the Riemann zeta function on very short segments of the critical line, based on the Riemann hypothesis.

Keywords: Riemann zeta function, the argument of the Riemann zeta function, Gram's law, critical line.

Bibliography: 36 titles.

Introduction

In 2001-2006, A.A. Karatsuba [1]-[4] obtained a series of lower estimates for the maximum of modulus of the Riemann zeta function $\zeta(s)$ in the circles of small radius lying in the critical strip $0 \leq \Re s \leq 1$, and on very short segments of the critical line $\Re s = 0.5$. These results gained further progress in [5]-[8].

In particular, it was proved in [4] that the function

$$F(T; H) = \max_{|t-T| \leq H} |\zeta(0.5 + it)|$$

satisfies the inequality

$$F(T; H) \geq \frac{1}{16} \exp \left\{ - \frac{5 \ln T}{6(\pi/\alpha - 1)(\cosh(\alpha H) - 1)} \right\}, \quad (1)$$

where α is any fixed number, $1 \leq \alpha < \pi$, $2 \leq \alpha H \leq \ln \ln H - c_1$, and $c_1 > 0$ is some absolute constant. Given $\varepsilon > 0$, it follows from (1) that for any $T \geq T_0(\varepsilon)$ and for $H \geq \pi^{-1}(1+\varepsilon) \ln \ln T - c_1$, the function $F(T; H)$ is bounded from below by some constant:

$$F(T; H) > c_2 = \frac{1}{16} \exp(-1.7\varepsilon^{-1}e^{c_1}) > 0.$$

In [4], A.A. Karatsuba posed the problem to prove the inequality $F(T; H) \geq 1$ for the values of H which are essentially smaller than $\ln \ln T$, namely, for $H \geq \ln \ln \ln T$.²⁾

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²⁾ If $\ln \ln T \ll H \leq 0.1T$, then the following estimate of R. Balasubramanian [9] holds true:

$$F(T; H) \gg \exp \left(\frac{3}{4} \sqrt{\frac{\ln H}{\ln \ln H}} \right).$$

This bound is supposed to be close to the best possible. Thus, the estimates of $F(T; H)$ for $0 < H \ll \ln \ln T$ are most interesting in this topic.

In this paper, we give a conditional solution of Karatsuba's problem, based on the Riemann hypothesis. Moreover, we prove that for arbitrary large fixed number $A \geq 1$ there exist positive constants T_0 and c_0 that depend on A and such that for any $T \geq T_0$ and $H = (1/\pi) \ln \ln \ln T + c_0$ the inequality $F(T; H) \geq A$ holds true (see Theorem 1).

The method used here is applicable to the estimation both of the maxima of the function

$$|\zeta(0.5 + it)| = \exp(\ln |\zeta(0.5 + it)|) = \exp(\Re \ln \zeta(0.5 + it)),$$

and the extremal values of the function

$$S(t) = \frac{1}{\pi} \arg \zeta(0.5 + it) = \frac{1}{\pi} \Im \ln \zeta(0.5 + it)$$

(for the definition and basic properties of the function $S(t)$, which is called the argument of the Riemann zeta-function on the critical line, see the survey [10]).

The estimates of maximum and minimum of the function $S(t)$ on very short segments of the variation of t hold the significant interest together with classical estimates of the values $\max_{T \leq t \leq 2T} (\pm S(t))$ belonging to A. Selberg [11] and K.-M. Tsang [12], [13]. Thus, the estimates of the form

$$\max_{|t-T| \leq H} (\pm S(t)) \geq f(H),$$

where

$$f(H) = \frac{1}{90\pi} \sqrt{\frac{\ln H}{\ln \ln H}}, \quad (\ln T)(\ln \ln T)^{-3/2} < H < T$$

and

$$f(H) = \frac{1}{900} \frac{\sqrt{\ln H}}{\ln \ln H}, \quad \sqrt{\ln \ln T} \leq H \leq (\ln T)(\ln \ln T)^{-3/2}$$

are obtained in [14] and [15], [16], respectively.

In this paper, we prove the existence of positive and negative values of the function $S(t)$ whose moduli exceed 3, on each segment of length $H = 0.8 \ln \ln \ln t + c_0$ (see Theorems 2-4). For comparison, we note that it appears in the process of calculation of first 200 billions zeros of $\zeta(s)$ on the critical line (S. Wedeniwski [17], 2003) that

$$\begin{aligned} |S(t)| &< 1 & \text{if } 7 < t < 280; \\ |S(t)| &< 2 & \text{if } 7 < t < 6\,820\,050; \\ |S(t)| &< 3 & \text{if } 7 < t < 16\,220\,609\,807. \end{aligned}$$

The first values of $S(t)$ which exceed 3 in modulus, are located in the neighborhoods of Gram points t_n (see §4) with indices $n = 53\,365\,784\,979$ и $n = 67\,976\,501\,145$ and are equal to 3.0214 and -3.2281 , respectively. At present time, no values of t such that $|S(t)| \geq 4$ are known.

Since the function $S(t)$ is "responsible" for the irregularity in the distribution of zeros of $\zeta(s)$, Theorems 3 and 4 imply some conditional results related the distribution of Gram's intervals $G_n = (t_{n-1}, t_n]$ which contain an "abnormal" (that is, unequal to 1) number of ordinates of zeros of $\zeta(s)$ (see Theorems 5, 6).

The paper ends with a proof of Theorem 7 that concerns the distribution of nonzero values of integer-valued function Δ_n introduced by A. Selberg [18] in connection with so-called Gram's law.

In this paper, we use the following notations: $\Lambda(n)$ denotes the von Mangoldt function, which is equal to $\ln p$ for prime p and $n = p^k$, $k = 1, 2, \dots$, and equal to zero otherwise; $\Lambda_1(n) = \Lambda(n)/\ln n$ ($n \geq 2$); $\cosh z = (e^z + e^{-z})/2$; $K_a(z) = \exp(-a \cosh z)$ ($a > 0$); \hat{f} denotes the Fourier transform of the function f , that is

$$\hat{f}(u) = \int_{-\infty}^{+\infty} f(x) e^{-iux} dx;$$

$\|\alpha\| = \min(\{\alpha\}, 1 - \{\alpha\})$ is the distance between α and the closest integer; $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \dots$ are primes indexed in ascending order; $\Omega(n)$ is the number of prime factors of n counted with multiplicity; $\theta, \theta_1, \theta_2, \dots$ are complex numbers, different in different formulae, whose absolute value does not exceed one. All other notations are explained in the text.

§1. Auxilliary assertions

In this section, we give some auxilliary lemmas.

LEMMA 1. *For any $m \geq 1$, the numbers*

$$1, \frac{1}{2\pi} \ln 2, \frac{1}{2\pi} \ln 3, \frac{1}{2\pi} \ln 5, \dots, \frac{1}{2\pi} \ln p_m$$

are linearly independent over the field of rationals.

PROOF. Let's assume the contrary. Then there exist the integers $k \geq 0, k_1, k_2, \dots, k_m$ not equal to zero simultaneously and such that

$$k + \frac{k_1}{2\pi} \ln 2 + \frac{k_2}{2\pi} \ln 3 + \dots + \frac{k_m}{2\pi} \ln p_m = 0,$$

or, which is the same,

$$k - \frac{1}{2\pi} \ln \frac{a}{b} = 0, \tag{2}$$

where a and b are coprime integers not equal to one simultaneously, whose prime factors do not exceed p_m . Exponentiating (2), we get

$$e^{2\pi k} = \frac{a}{b}. \tag{3}$$

If $k = 0$ then (3) contradicts the fundamental theorem of arithmetics. If $k \geq 1$ then e^π appears to be the root of polynomial $bz^{2k} - a$. This is impossible in view of transcendence of e^π (see for example [19, §2.4]). These contradictions prove the lemma.

LEMMA 2. *The estimate $|\hat{K}_a(\lambda)| \leq \kappa e^{-b|\lambda|}$ holds for any real λ with*

$$\kappa = \kappa(a, b) = 2 \int_0^{+\infty} \exp(-a(\cos b) \cosh u) du,$$

where b is any number with the condition $0 < b < \pi/2$.

The proof of this statement repeats almost verbatim that of Lemma 4 in [20].

LEMMA 3. Suppose that λ is real and satisfies the condition $|\lambda| \geq a\sqrt{2}$. Then the following relation holds:

$$\widehat{K}_a(\lambda) = \frac{2\sqrt{2\pi}}{\sqrt[4]{\lambda^2 - a^2}} \exp\left(-\frac{\pi|\lambda|}{2}\right) (\cos g_a(\lambda) + r_a(\lambda)),$$

where

$$g_a(\lambda) = \sqrt{\lambda^2 - a^2} - |\lambda| \ln\left(\frac{|\lambda|}{a} + \frac{\sqrt{\lambda^2 - a^2}}{a}\right) + \frac{\pi}{4}, \quad |r_a(\lambda)| \leq c_a |\lambda|^{-0.1},$$

and

$$c_a = \begin{cases} 9.3, & \text{if } a \geq 1/\sqrt{2}, \\ 8.2a^{-0.4}, & \text{if } 0 < a < 1/\sqrt{2}. \end{cases}$$

PROOF. Without loss of generality, we assume that $\lambda > 0$. Let's take an arbitrary $R > 1$ and denote by Γ_R the contour of rectangle with vertices at the points $\pm R$, $\pm R - \pi i/2$, traversed counterclockwise. The application of Cauchy's residue theorem yields

$$\int_{\Gamma_R} K_a(z) e^{-i\lambda z} dz = \sum_{k=1}^4 I_k = 0,$$

where I_1, I_3 are integrals along the upper and lower sides of contour and I_2, I_4 are integrals over lateral sides.

Further, it is easy to note that

$$\begin{aligned} -I_1 &= \int_{-R}^R K_a(u) e^{-i\lambda u} du, \\ I_3 &= \int_{-R}^R K_a\left(u - \frac{\pi i}{2}\right) e^{-i\lambda\left(u - \frac{\pi i}{2}\right)} du = e^{-\frac{\pi\lambda}{2}} \int_{-R}^R e^{i\varphi_a(u)} du, \end{aligned}$$

where $\varphi_a(u) = a \sinh u - \lambda u$. Let us put $z = R - \pi it/2$, where $0 \leq t \leq 1$. Since $|K_a(z)| = e^{-a \cosh(R) \cos(\pi t/2)}$, we get:

$$\begin{aligned} |I_4| &\leq \frac{\pi}{2} \int_0^1 e^{-a \cosh(R) \cos(\pi t/2)} dt = \frac{\pi}{2} \int_0^1 e^{-a \cosh(R) \sin(\pi t/2)} dt \leq \frac{\pi}{2} \int_0^1 e^{-at \cosh(R)} dt \leq \\ &\leq \frac{\pi}{2a \cosh R}. \end{aligned}$$

The same bound is valid for the integral I_2 . Hence,

$$\int_{-R}^R K_a(u) e^{-i\lambda u} du = e^{-\frac{\pi\lambda}{2}} \int_{-R}^R e^{i\varphi_a(u)} du + \frac{\pi\theta}{a \cosh R}.$$

Letting R tend to infinity, we obtain:

$$\widehat{K}_a(\lambda) = e^{-\frac{\pi\lambda}{2}} \int_{-\infty}^{+\infty} e^{i\varphi_a(u)} du = 2e^{-\frac{\pi\lambda}{2}} \Re j_a(\lambda), \quad j_a(\lambda) = \int_0^{+\infty} e^{i\varphi_a(u)} du.$$

The derivative $\varphi'_a(u)$ has a unique zero on the ray of integration at a point

$$u_a = \operatorname{arccosh} \frac{\lambda}{a} = \ln \left(\frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1} \right).$$

Setting $u = u_a + v$, where $-u_a \leq v < +\infty$ and noting that

$$\varphi_a(u) = a(\sinh u_a \cosh v + \cosh u_a \sinh v) - \lambda(u_a + v) = -\lambda u_a + \lambda \psi_a(v),$$

where $\psi_a(v) = \alpha \cosh v + \sinh v - v$, $\alpha = \sqrt{1 - (a/\lambda)^2}$, we find that

$$j_a(\lambda) = e^{-i\lambda u_a} \int_{-u_a}^{+\infty} e^{i\lambda \psi_a(v)} dv.$$

Suppose that δ satisfies the condition $0 < \delta < \min(1, u_a, \lambda^{-1/3})$. Then we represent $j_a(\lambda)$ as the sum

$$e^{-i\lambda u_a} \left(\int_{-\delta}^{\delta} + \int_{-u_a}^{-\delta} + \int_{\delta}^{+\infty} \right) e^{i\lambda \psi_a(v)} dv = e^{-i\lambda u_a} (j_1 + j_2 + j_3).$$

We have

$$\psi_a(v) = \psi_a(0) + \psi'_a(0)v + \psi''_a(0) \frac{v^2}{2} + \psi_a^{(3)}(\xi) \frac{v^3}{6}$$

for $|v| \leq \delta$, where ξ lies between 0 and v . Since

$$\psi'_a(v) = \alpha \sinh v + \cosh v - 1, \quad \psi''_a(v) = \alpha \cosh v + \sinh v, \quad \psi_a^{(3)}(v) = \alpha \sinh v + \cosh v,$$

then $\psi_a(0) = \psi''_a(0) = \alpha$, $\psi'_a(0) = 0$, and

$$|\psi_a^{(3)}(\xi)| = |\alpha \sinh \xi + \cosh \xi| \leq \sinh |\xi| + \cosh \xi = e^{|\xi|} \leq e^\delta < e.$$

Hence,

$$\lambda \psi_a(v) = \mu + \mu \frac{v^2}{2} + e\lambda \frac{\theta v^3}{6}, \quad \mu = \alpha\lambda = \sqrt{\lambda^2 - a^2}.$$

Let us define $\varrho(v)$ by the relation $\exp(i e \theta \lambda v^3 / 6) = 1 + \varrho(v)$. Thus we get

$$\begin{aligned} |\varrho(v)| &= \left| \frac{ie\lambda}{6} \theta v^3 + \frac{1}{2!} \left(\frac{ie\lambda}{6} \theta v^3 \right)^2 + \frac{1}{3!} \left(\frac{ie\lambda}{6} \theta v^3 \right)^3 + \dots \right| \leq \\ &\leq \frac{e\lambda}{6} |v|^3 \left(1 + \frac{1}{2!} \frac{e}{6} + \frac{1}{3!} \left(\frac{e}{6} \right)^2 + \dots \right) = (e^{e/6} - 1) \lambda |v|^3 < \frac{3\lambda}{5} |v|^3. \end{aligned}$$

Therefore,

$$\begin{aligned} j_1 &= \int_{-\delta}^{\delta} \exp\left(i\mu + \frac{i\mu v^2}{2}\right) (1 + \varrho(v)) dv = e^{i\mu} \int_{-\delta}^{\delta} \exp\left(\frac{i\mu v^2}{2}\right) dv + 2\theta_1 \int_0^{\delta} \frac{3\lambda}{5} v^3 dv = \\ &= 2e^{i\mu} \int_0^{\delta} \exp\left(\frac{i\mu v^2}{2}\right) dv + \frac{3\theta_1}{10} \lambda \delta^4 = e^{i\mu} \sqrt{\frac{2}{\mu}} \int_0^{\frac{\mu}{2}\delta^2} \frac{e^{iw} dw}{\sqrt{w}} + \frac{3\theta_1}{10} \lambda \delta^4. \end{aligned}$$

Replacing the last integral by improper one and noting that

$$\int_0^{+\infty} \frac{e^{iw} dw}{\sqrt{w}} = e^{\pi i/4} \sqrt{\pi}, \quad \left| \int_u^{+\infty} \frac{e^{iw} dw}{\sqrt{w}} \right| \leq \frac{2}{\sqrt{u}},$$

we find that

$$j_1 = e^{i\mu} \sqrt{\frac{2}{\mu}} \left(\sqrt{\pi} e^{\pi i/4} + \frac{2\theta_2 \sqrt{2}}{\sqrt{\mu} \delta^2} \right) + \frac{3\theta_1}{10} \lambda \delta^4 = \sqrt{\frac{2\pi}{\mu}} e^{i(\mu + \pi/4)} + \theta_3 \left(\frac{4}{\mu \delta} + \frac{3\lambda \delta^4}{10} \right)$$

for any $u > 0$. Further, the integration by parts in j_2 yields:

$$j_2 = \frac{1}{i\lambda} \left(\frac{e^{i\lambda\psi_a(-\delta)}}{\psi'_a(-\delta)} - \frac{e^{i\lambda\psi_a(-u_a)}}{\psi'_a(-u_a)} - \int_{-u_a}^{-\delta} e^{i\lambda\psi_a(v)} d \frac{1}{\psi'_a(v)} \right)$$

and hence

$$|j_2| \leq \frac{1}{\lambda} \left(\frac{1}{|\psi'_a(-\delta)|} + \frac{1}{|\psi'_a(-u_a)|} + \int_{-u_a}^{-\delta} \left| d \frac{1}{\psi'_a(v)} \right| \right).$$

Since

$$\alpha = \frac{\sqrt{\lambda^2 - a^2}}{\lambda} = \frac{\sinh u_a}{\cosh u_a} = \tanh u_a,$$

then the derivative $\psi''_a(v) = \cosh v (\alpha + \tanh v)$ is positive for $v > -u_a$. Thus, the function $1/\psi'_a(v)$ decreases for $v > -u_a$. Hence,

$$\begin{aligned} |j_2| &\leq \frac{1}{\lambda} \left(\frac{1}{|\psi'_a(-\delta)|} + \frac{1}{|\psi'_a(-u_a)|} - \int_{-u_a}^{-\delta} d \frac{1}{\psi'_a(v)} \right) = \\ &= \frac{1}{\lambda} \left(\frac{1}{|\psi'_a(-\delta)|} + \frac{1}{|\psi'_a(-u_a)|} - \frac{1}{\psi'_a(-\delta)} + \frac{1}{\psi'_a(-u_a)} \right). \end{aligned}$$

Since $\psi'_a(0) = 0$, then $\psi'_a(v) < 0$ for negative v and therefore

$$j_2 \leq \frac{2}{\lambda |\psi'_a(-\delta)|}.$$

Further, we have

$$|\psi'_a(-\delta)| = |\alpha \sinh \delta - \cosh \delta + 1| = 2 \sinh \frac{\delta}{2} \left| \alpha \cosh \frac{\delta}{2} - \sinh \frac{\delta}{2} \right| > \delta \left| \alpha \cosh \frac{\delta}{2} - \sinh \frac{\delta}{2} \right|.$$

Since $\lambda \geq a\sqrt{2}$, then $\alpha \geq 1/\sqrt{2}$ and hence

$$\begin{aligned} \alpha \cosh \frac{\delta}{2} - \sinh \frac{\delta}{2} &\geq \frac{1}{\sqrt{2}} \cosh \frac{\delta}{2} - \sinh \frac{\delta}{2} \geq \frac{1}{\sqrt{2}} \left(1 + \frac{1}{2!} \left(\frac{\delta}{2} \right)^2 + \frac{1}{4!} \left(\frac{\delta}{2} \right)^4 + \dots \right) - \\ &\quad - \left(\frac{\delta}{2} + \frac{1}{3!} \left(\frac{\delta}{2} \right)^3 + \frac{1}{5!} \left(\frac{\delta}{2} \right)^5 + \dots \right) > \frac{1}{\sqrt{2}} - \frac{\delta}{2} > \frac{1}{5}. \end{aligned}$$

Finally we get:

$$|\psi'_a(-\delta)| > \frac{\delta}{5}, \quad |j_2| < \frac{10}{\lambda\delta} < \frac{10}{\mu\delta}.$$

The proof of the inequality $|j_3| \leq 2(\lambda\psi'_a(\delta))^{-1}$ is just the same. By the relations $\psi'_a(\delta) = \alpha \sinh \delta + \cosh \delta - 1 > \alpha\delta \geq \delta/\sqrt{2}$, it implies that

$$|j_3| \leq \frac{2\sqrt{2}}{\lambda\delta} < \frac{3}{\mu\delta}.$$

Therefore,

$$j_1 + j_2 + j_3 = \sqrt{\frac{2\pi}{\mu}} e^{i(\mu+\pi/4)} + r_1,$$

where

$$|r_1| \leq \frac{4}{\mu\delta} + \frac{3\lambda\delta^4}{10} + \frac{10}{\mu\delta} + \frac{3}{\mu\delta} = \frac{17}{\mu\delta} + \frac{3\lambda\delta^4}{10}.$$

Thus we conclude that

$$j_a(\lambda) = \sqrt{\frac{2\pi}{\mu}} e^{i(\mu+\pi/4-\lambda u_a)} (1 + r_2),$$

where

$$|r_2| \leq \sqrt{\frac{\mu}{2\pi}} \left(\frac{17}{\mu\delta} + \frac{3\lambda\delta^4}{10} \right) \leq \frac{1}{\sqrt{\pi}} \left(\frac{17}{\sqrt[4]{2}\delta\sqrt{\lambda}} + \frac{3\lambda^{3/2}\delta^4}{10\sqrt{2}} \right).$$

If $a\sqrt{2} \geq 1$, we put $\delta = (7/8)\lambda^{-2/5}$. Since $\lambda \geq a\sqrt{2} \geq 1$, the inequalities $\delta < 1$, $\delta < \lambda^{-1/3}$ are obvious. Moreover,

$$u_a = \ln \left(\frac{\lambda}{a} + \sqrt{\left(\frac{\lambda}{a} \right)^2 - 1} \right) \geq \ln(\sqrt{2} + 1) > \frac{7}{8} \geq \delta,$$

and hence $\delta < \min(1, \lambda^{-1/3}, u_a)$. Thus, we have in this case:

$$|r_2| \leq \frac{1}{\sqrt{\pi}} \left(\frac{8 \cdot 17}{7\sqrt[4]{2}} + \frac{3}{10\sqrt{2}} \left(\frac{7}{8} \right)^4 \right) \lambda^{-1/10} < 9.3\lambda^{-0.1}.$$

If $a\sqrt{2} < 1$ then we put $\delta = (a/\lambda)^{2/5}$. Then the inequality $\lambda \geq a\sqrt{2}$ implies that

$$\delta \leq (1/\sqrt{2})^{2/5} < 1, \quad a^6 < a \left(\frac{1}{\sqrt{2}} \right)^5 = \frac{a\sqrt{2}}{8} \leq \frac{\lambda}{8} < \lambda,$$

and $a^{2/5} < \lambda^{1/15} = \lambda^{2/5-1/3}$. Thus, $\delta < \lambda^{-1/3}$. Finally, since the inequality $x^{-2/5} < \ln(x + \sqrt{x^2 - 1})$ holds for any $x \geq \sqrt{2}$, we find $\delta < u_a$. Therefore, in this case, the inequality $\delta < \min(1, \lambda^{-1/3}, u_a)$ is also valid. Thus we obtain

$$|r_2| \leq \frac{1}{\sqrt{\pi}} \left(\frac{17}{\sqrt[4]{2}} + \frac{3a^2}{10\sqrt{2}} \right) a^{-2/5} \lambda^{-1/10} < 8.2a^{-0.4} \lambda^{-0.1}.$$

Finally we get

$$\begin{aligned} \widehat{K}_a(\lambda) &= 2e^{-\pi\lambda/2} \sqrt{\frac{2\pi}{\mu}} \Re \left(e^{i(\mu - \lambda u_a + \pi/4)} (1 + r_2) \right) = \\ &= 2\sqrt{\frac{2\pi}{\mu}} e^{-\pi\lambda/2} (\cos(\mu - \lambda u_a + \pi/4) + r), \end{aligned}$$

where $|r| \leq c_a \lambda^{-0.1}$ is such that $c_a = 9.3$ for $a\sqrt{2} \geq 1$ and $c_a = 8.2a^{-0.4}$ for $0 < a\sqrt{2} < 1$. The lemma is proved.

COROLLARY. *Under the conditions of Lemma 3, the following inequality holds*

$$|\widehat{K}_a(\lambda)| < \kappa_a \frac{e^{-\pi|\lambda|/2}}{\sqrt{|\lambda|}},$$

where $\kappa_a = 61.5$ for $a\sqrt{2} \geq 1$ and $\kappa_a = 54.1a^{-0.4}$ for $0 < a\sqrt{2} < 1$.

PROOF. The inequality of Lemma 3 together with the condition $|\lambda| \geq a\sqrt{2}$ imply that

$$|\widehat{K}_a(\lambda)| < \frac{2\sqrt{2\pi}}{\sqrt{|\lambda|}} \frac{e^{-\pi|\lambda|/2}}{\sqrt[4]{1 - (a/\lambda)^2}} (1 + r) \leq \frac{2^{7/4}\sqrt{\pi}}{\sqrt{|\lambda|}} e^{-\pi|\lambda|/2} (1 + r),$$

where $r = c_a |\lambda|^{-1/10}$. Using the above expressions for c_a , we get the desired bound.

LEMMA 4. *Suppose that the function $f(z)$ is analytical in the strip $|\Im z| \leq 0.5 + \alpha$, where it satisfies the inequality $|f(z)| \leq c(|z| + 1)^{-(1+\beta)}$ with some positive β and c . Then the identity*

$$\begin{aligned} \int_{-\infty}^{+\infty} f(u) \ln \zeta(0.5 + i(t + u)) du &= \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{f}(\ln n) + \\ &+ 2\pi \left(\sum_{\beta > 0.5} \int_0^{\beta-0.5} f(\gamma - t - iv) dv - \int_0^{0.5} f(-t - iv) dv \right), \quad (4) \end{aligned}$$

holds for any t , where $\varrho = \beta + i\gamma$ in the last sum runs through all complex zeros of $\zeta(s)$ to the right from the critical line.

This assertion goes back to A. Selberg (see for example [11, Lemma 16]). In [10, Ch. II, §2], [12], there are some variants of this lemma, where $f(z)$ satisfies slightly different conditions. These proofs can be easily adopted to the case under considering.

LEMMA 5. *If the Riemann hypothesis is true then the relation*

$$\int_{-\infty}^{+\infty} K_a(\pi u) \ln \zeta(0.5 + i(t + u)) du = \frac{1}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) - 2\pi \int_0^{0.5} K_a(\pi t + \pi iv) dv \quad (5)$$

holds for any real t .

PROOF. We take an arbitrary δ such that $0 < \delta < 10^{-6}$ and set $z = x + iy$, $f(z) = K_a((\pi - \delta)z)$, $\alpha = \delta/(4\pi)$. Since the inequalities

$$\cos(\pi - \delta)y \geq \cos\{(\pi - \delta)(0.5 + \alpha)\} > \sin \frac{\delta}{4} \geq 2\alpha,$$

hold for any y such that $|y| \leq 0.5 + \alpha$, then we have

$$|f(z)| = e^{-a \cosh(\pi - \delta)x \cos(\pi - \delta)y} \leq e^{-2a\alpha \cosh(\pi - \delta)x} \leq c(|z| + 1)^{-(1+\beta)}.$$

for a suitable constants $\beta = \beta(\alpha)$, $c = c(\alpha)$ and for any x .

The application of Lemma 4 yields:

$$\begin{aligned} \int_{-\infty}^{+\infty} K_a((\pi - \delta)u) \ln \zeta(0.5 + i(t + u)) du &= \\ &= \frac{1}{\pi - \delta} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi - \delta}\right) - 2\pi \int_0^{0.5} K_a((\pi - \delta)(t + iv)) dv. \end{aligned} \quad (6)$$

Let us take

$$N = \left\lceil \frac{1}{\delta^2} \left(\ln \frac{1}{\delta} \right)^{-1} \right\rceil + 1$$

and suppose δ to be so small that $N > N_0 = e^{\pi a \sqrt{2}}$. Now we split the sum in (6) to the sums C_1, C_2 and C_3 corresponding to the intervals $n > N$, $N_0 < n \leq N$ и $n \leq N_0$, respectively. Using the Corollary of Lemma 3 with $\lambda = (1/\pi) \ln n \geq a\sqrt{2}$, we obtain

$$|C_1| \leq \frac{1}{\pi - \delta} \sum_{n > N} \frac{\Lambda_1(n)}{\sqrt{n}} 61.5 \sqrt{\frac{\pi - \delta}{\ln n}} \exp\left(-\frac{\pi}{2} \frac{\ln n}{\pi - \delta}\right) \leq \frac{61.5}{\sqrt{\pi - \delta}} \sum_{n > N} \frac{\Lambda(n)}{n(\ln n)^{3/2}}.$$

The application of Abel's summation formula together with the bound

$$\psi(u) = \sum_{n \leq u} \Lambda(n) \leq c_1 u, \quad c_1 = 1.03883 \quad (7)$$

(see [23, Th. 12]), which is valid for any $u > 0$, yields:

$$\begin{aligned} \sum_{n > N} \frac{\Lambda(n)}{n(\ln n)^{3/2}} &= - \int_N^{+\infty} (\psi(u) - \psi(N)) d \frac{1}{(\ln u)^{3/2}} \leq -c_1 \int_N^{+\infty} u d \frac{1}{(\ln u)^{3/2}} = \\ &= c_1 \left(\frac{2}{\sqrt{\ln N}} + \frac{1}{(\ln N)^{3/2}} \right). \end{aligned}$$

Using the inequalities $\ln N \geq \ln(1/\delta)$ and $0 < \delta < 10^{-6}$, we get the estimate

$$|C_1| \leq \frac{123c_1}{\sqrt{\pi - \delta}} \frac{1}{\sqrt{\ln(1/\delta)}} \left(1 + \frac{1}{2\ln(1/\delta)}\right) < \frac{75}{\sqrt{\ln(1/\delta)}}.$$

Similarly,

$$\left| \frac{1}{\pi} \sum_{n > N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \right| < \frac{74.9}{\sqrt{\ln(1/\delta)}}.$$

Thus we get:

$$C_1 = \frac{1}{\pi} \sum_{n > N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) + \frac{149.9}{\sqrt{\ln(1/\delta)}}.$$

Further, we represent C_2 as

$$\frac{1}{\pi - \delta} \sum_{N_0 < n \leq N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) - \frac{1}{\pi} \sum_{N_0 < n \leq N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} d_n,$$

where

$$d_n = \widehat{K}_a\left(\frac{\ln n}{\pi}\right) - \widehat{K}_a\left(\frac{\ln n}{\pi - \delta}\right) = \int_{-\infty}^{+\infty} K_a(u) (e^{-i\varphi_1} - e^{-i\varphi_2}) du,$$

$$\varphi_1 = \frac{u \ln n}{\pi}, \quad \varphi_2 = \frac{u \ln n}{\pi - \delta}.$$

Since

$$|e^{-i\varphi_1} - e^{-i\varphi_2}| = 2 \left| \sin \frac{\varphi_1 - \varphi_2}{2} \right| \leq |\varphi_1 - \varphi_2| = \frac{\delta |u| \ln n}{\pi(\pi - \delta)},$$

we obtain:

$$|d_n| \leq \frac{\delta |u| \ln n}{\pi(\pi - \delta)} \int_{-\infty}^{+\infty} |u| e^{-\cosh(\pi u)} du < 0.01 \delta \ln n.$$

Using the bound (7) again, we get:

$$\begin{aligned} \left| \sum_{N_0 < n \leq N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{it} d_n \right| &\leq 0.01 \delta \sum_{N_0 < n \leq N} \frac{\Lambda(n)}{\sqrt{n}} \leq \\ &\leq 0.01 \delta \left(\frac{\psi(N)}{\sqrt{N}} + \frac{1}{2} \int_1^N \frac{\psi(u)}{u^{3/2}} du \right) \leq 0.02 c_1 \delta \sqrt{N} < \frac{0.1}{\sqrt{\ln(1/\delta)}}, \end{aligned}$$

and hence

$$C_2 = \frac{1}{\pi - \delta} \sum_{N_0 < n \leq N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) + \frac{0.1\theta}{\sqrt{\ln(1/\delta)}}.$$

Finally, the error arising from the replacement of $\pi - \delta$ by π in the last expression does not exceed

$$\frac{\delta}{\pi(\pi - \delta)} \sum_{N_0 < n \leq N} \frac{\Lambda_1(n)}{\sqrt{n}} \left| \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \right| \leq \frac{61.5 \delta \sqrt{\pi}}{\pi(\pi - \delta)} \sum_{n \leq N} \frac{\Lambda(n)}{n(\ln n)^{3/2}} < 25\delta$$

in modulus. Therefore,

$$C_2 = \frac{1}{\pi} \sum_{N_0 < n \leq N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) + \theta\left(25\delta + \frac{0.1}{\sqrt{\ln(1/\delta)}}\right).$$

Thus, the relation (6) takes the form

$$\begin{aligned} \int_{-\infty}^{+\infty} K_a((\pi - \delta)u) \ln \zeta(0.5 + i(t + u)) du &= \\ &= \frac{1}{\pi - \delta} \sum_{n \leq N_0} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi - \delta}\right) + \frac{1}{\pi} \sum_{n > N_0} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) - \\ &\quad - 2\pi \int_0^{0.5} K_a((\pi - \delta)(t + iv)) dv + \theta\left(25\delta + \frac{150}{\sqrt{\ln(1/\delta)}}\right). \end{aligned} \quad (8)$$

The integrals in both sides of (8) and the sum C_3 over $n \leq N_0$ are continuous functions of δ , $0 \leq \delta \leq 10^{-6}$. Tending δ to zero, we lead to the desired statement. The Lemma is proved.

§2. Basic lemma

The classical ‘Dirichlet’s approximation theorem’ asserts that for any fixed vector $(\alpha_1, \dots, \alpha_m)$ with real components and for any arbitrary small ε , $0 < \varepsilon < 0.5$, the interval $(1, c)$, $c = \varepsilon^{-m}$, contains a number t such that the following inequalities hold: $\|t\alpha_j\| < \varepsilon$, $j = 1, \dots, m$.

Its standard proof (see, for example, [21, Appendix, §9, Theorem 4]) does not allow one to state the existence of a number t with the above property on every interval of the type $(T, T + c_1)$, where $c_1 > 0$ is a constant depending only on the tuple $(\alpha_1, \dots, \alpha_m)$ and ε .

In this section, we prove the analogue of Dirichlet’s theorem which is free of the above disadvantage³⁾. However, we note that the replacement of the interval $(1, c)$ by an arbitrary interval $(T, T + c_1)$ leads to the loss of generality (the condition of linear independence of numbers $1, \alpha_1, \dots, \alpha_m$ over the field \mathbb{Q} of the rationals appears) and to inefficiency of the constant $c_1 = c_1(\alpha_1, \dots, \alpha_m; \varepsilon)$. The last fact is a reason of the inefficiency of the constants c_0 and T_0 in Theorems 1-6 (c_0 and N_0 in Theorem 7, respectively) and of the impossibility of replacement the value A in Theorem 1 by some increasing function of the parameter T .

LEMMA 6. *For any vector $\bar{\alpha} = (1, \alpha_1, \dots, \alpha_n)$ whose components are linearly independent over the rationals and for any ε , $0 < \varepsilon < 0.5$, there exists a constant $c = c(\bar{\alpha}, \varepsilon)$ such that any interval of length c contains at least one value t such that the following inequalities hold: $\|t\alpha_j\| < \varepsilon$, $j = 1, \dots, n$.*

³⁾The author sincerely appreciates O.N. German and N.G. Moshchevitin who kindly communicated him the idea of the proof of Lemma 6.

PROOF. We precede the proof by some remarks.

REMARK 1. Let l be the line in \mathbb{R}^{n+1} which is parallel to vector $\bar{\alpha}$ and passing through the origin, and let $X = (x_0, x_1, \dots, x_n)$ be a point. Then the distance $d = d(X)$ between X and l is given by a formula

$$d = \frac{1}{|\bar{\alpha}|} \sqrt{\sum_{0 \leq i < j \leq n} \Delta_{ij}^2}, \quad \text{where} \quad |\bar{\alpha}| = \sqrt{1 + \sum_{1 \leq j \leq n} \alpha_j^2}, \quad (9)$$

and Δ_{ij} is a minor of matrix

$$\begin{pmatrix} 1 & \alpha_1 & \dots & \alpha_n \\ x_0 & x_1 & \dots & x_n \end{pmatrix},$$

generated by columns i and j . Suppose that the lattice point $M = (m_0, m_1, \dots, m_n)$ satisfies the inequality $d(M) < \varepsilon_1 = \varepsilon|\alpha|^{-1}$. Then

$$\sum_{0 \leq i < j \leq n} \Delta_{ij}^2 < \varepsilon^2$$

and therefore

$$|\Delta_{01}| = |\alpha_1 m_0 - m_1| < \varepsilon, \quad \dots, \quad |\Delta_{0n}| = |\alpha_n m_0 - m_n| < \varepsilon. \quad (10)$$

In view of condition $0 < \varepsilon < 0.5$, the inequalities (10) imply that $\|\alpha_j t\| < \varepsilon$ for any j , $1 \leq j \leq n$, and for $t = m_0$.

Thus, it suffices to prove the existence of the infinite sequence of points M_j of the lattice \mathbb{Z}^{n+1} such that the distance between any neighbouring points M_j and M_{j+1} is bounded from above by some constant depending only on $\bar{\alpha}$ and ε .

REMARK 2. Let us put

$$\delta = \frac{\varepsilon_1}{n+1} = \frac{\varepsilon|\bar{\alpha}|^{-1}}{n+1}$$

and denote by C_δ the infinite cylinder of radius δ with axis l in \mathbb{R}^{n+1} . Suppose that there exist the points $K_1, \dots, K_{n+1} \in \mathbb{Z}^{n+1}$ inside C_δ such that the vectors $\bar{v}_j = \overrightarrow{OK_j}$, $j = 1, \dots, n+1$ are linearly independent. Then $\bar{v}_1, \dots, \bar{v}_{n+1}$ generate an integer lattice \mathcal{L} in \mathbb{R}^{n+1} with fundamental domain Π , where Π is a parallelepiped spanned on $\bar{v}_1, \dots, \bar{v}_{n+1}$.

It is known that any shift $\Pi + \bar{\xi}$ of the parallelepiped Π to vector $\bar{\xi} \in \mathbb{R}^{n+1}$ contains a point of lattice \mathcal{L} which is also a point of lattice \mathbb{Z}^{n+1} . Further, Π is obviously contained in a cylinder $C_{\varepsilon_1} = (n+1)C_\delta$ of radius $(n+1)\delta = \varepsilon_1$ which is coaxial to C_δ .

Hence, any shift $\Pi + \bar{\xi}$ to vector $\bar{\xi}$ parallel to $\bar{\alpha}$ is fully contained inside C_{ε_1} . At the same time, this shift contains some lattice point $M(\bar{\xi})$.

Choosing the vectors $\bar{\xi}_j$ in such way that the shifts $\Pi + \bar{\xi}_j$ have no pairwise intersections, we find the desired infinite sequence of lattice points $M_j = M(\bar{\xi}_j)$ (see Fig. 1).

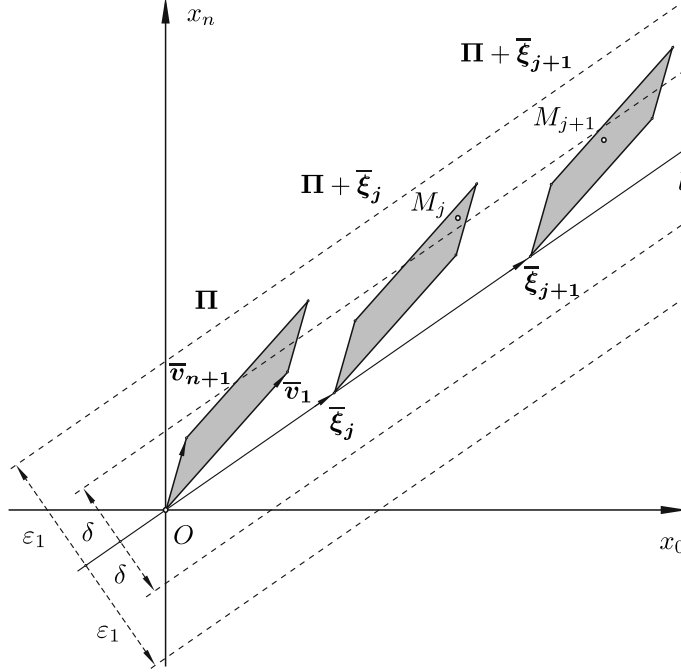


Fig. 1. Any shift $\Pi + \bar{\xi}_j$ of the parallelepiped Π contains a point M_j of the lattice \mathbb{Z}^{n+1} .

Thus, taking $\bar{\xi}_j = jc_0 \bar{\alpha}$, $j = 0, \pm 1, \pm 2, \dots$, where $c_0 = 2(|\bar{v}_1| + \dots + |\bar{v}_{n+1}|)$ is duplicated sum of lengths of edges of the parallelepiped Π originating from the same vertex, one can check that the first coordinate of vertex $\bar{\xi}_j$ of $\Pi + \bar{\xi}_j$, which is equal to jc_0 , differs from the first coordinate $m_0^{(j)}$ of lattice point M_j for at most $|\bar{v}_1| + \dots + |\bar{v}_{n+1}| = 0.5c_0$. In view of Remark 1, each of these first coordinates satisfies the series of inequalities $\|\alpha_i m_0^{(j)}\| < \varepsilon$, $i = 1, \dots, n+1$. Since

$$|m_0^{(j)} - m_0^{(j+1)}| \leq (j+1)c_0 + 0.5c_0 - (jc_0 - 0.5c_0) = 2c_0,$$

it appears that any interval of the type $(\tau, \tau + 3c_0)$ contains a point of sequence $m_0^{(j)}$, $j = 0, \pm 1, \pm 2, \dots$

Thus, it suffices to prove that any cylinder C_δ with axis l contains $n+1$ linearly independent vectors of the lattice \mathbb{Z}^{n+1} .

Now let us prove the main assertion. First we show that C_δ contains an infinite set of lattice points.

The line l does not contain lattice points different from the origin O . In the opposite case, we have $d(K) = 0$, $k_0 \neq 0$ for such point $K = (k_0, k_1, \dots, k_n) \in \mathbb{Z}^{n+1}$. Hence $\Delta_{0j} = \alpha_j k_0 - k_j = 0$ for any $j = 1, \dots, n$ and therefore, $\alpha_j = k_j/k_0 \in \mathbb{Q}$. But this contradicts the linear independence of $1, \alpha_1, \dots, \alpha_n$ over the rationals.

Let Ω_n be an n -dimensional hyperplane passing through the origin O perpendicularly to the axis l . Then an n -dimensional volume V_1 of a sphere arising in the intersection of the cylinder C_δ with the hyperplane Ω_n is equal to $V_1 = c(n)\delta^n$, where $c(n) = \pi^{n/2}\Gamma^{-1}(n/2 + 1)$. Now let us define H_1 by the relation $H_1 V_1 = 2^{n-1}$ and consider an $(n+1)$ -dimensional cylinder T_1 of height $2H_1$ which arises from C_δ after the section by two hyperplanes parallel to Ω_n which are distant to H_1 from the origin.

Since the volume of such cylinder is equal to $2H_1V_1 = 2^n$, Minkowski's convex body theorem (see for example [22, §5]) implies that this cylinder contains a lattice point N_1 different from the origin O .

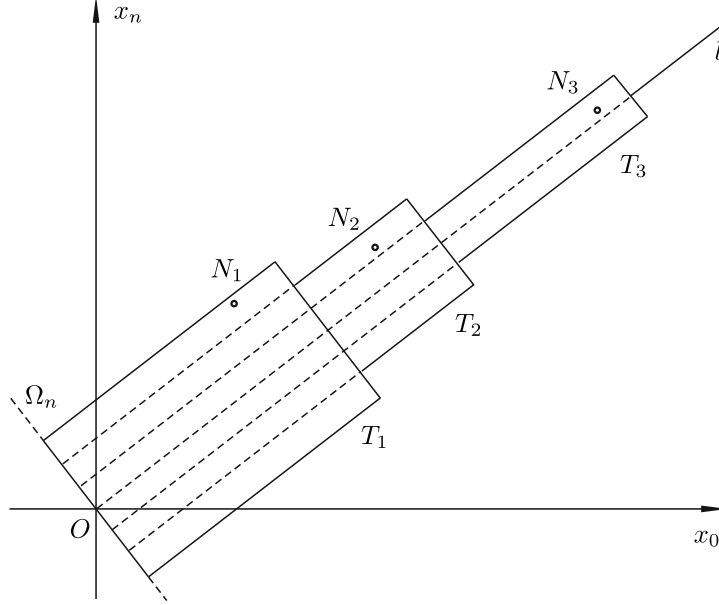


Fig. 2. An infinite sequence of lattice points N_j .

Without loss of generality, we assume that N_1 is the closest point to l among the lattice points of the cylinder T_1 which differs from the origin O . In view of the above remark, N_1 does not lie on l , so we have $d(N_1) > 0$.

Further, let us take $\delta_2 = 0.5d(N_1)$ and define H_2 by the relations $H_2V_2 = 2^{n-1}$, $V_2 = c(n)\delta_2^n$. Applying the same arguments to the cylinder T_2 of radius δ_2 and height $2H_2$, which is symmetrical with respect to the origin and coaxial to T_1 , we find a lattice point N_2 inside it, which is different from the origin O and closest to l among the lattice points of T_2 . Since $d(N_2) \leq \delta_2 < d(N_1)$, the point N_2 differs from N_1 . In view of symmetry both of T_1 and T_2 with respect to O , we assume that N_1 and N_2 lie in the same half-space with respect to the hyperplane Ω_n .

Taking $\delta_3 = 0.5d(N_2)$, $H_3V_3 = 2^{n-1}$, $V_3 = c(n)\delta_3^n$, we construct in the same way the cylinder T_3 of radius δ_3 and height $2H_3$ and find a lattice point N_3 inside it, which differs from O , N_1 , N_2 and lying in the same half-space with respect to Ω_n .

Proceeding this process further, we finally get an infinite sequence of different points N_j of the lattice \mathbb{Z}^{n+1} containing in the same half of the cylinder C_δ with respect to secant hyperplane Ω_n and satisfying the condition $0 < d(N_{j+1}) \leq 0.5d(N_j)$, $j = 1, 2, 3, \dots$

Now we prove the existence of $n + 1$ linearly independent vectors among the infinite set $\overrightarrow{ON_j}$, $j = 1, 2, 3, \dots$

Let's assume the contrary. Suppose that the maximal number s of linearly independent vectors from this set does not exceed n . Let $\bar{u}_1, \dots, \bar{u}_s \in \mathbb{Z}^{n+1}$ be such vectors and let ω_s be the s -dimensional hyperplane spanned on it.

Then the intersection of ω_s and C_δ contains an infinite sequence of points N_j of lattice \mathbb{Z}^{n+1} . Hence, this intersection is unbounded. But the intersection of ω_s and C_δ is

unbounded if and only if the hyperplane ω_s is parallel to the line l or contains it (see Fig. 3).

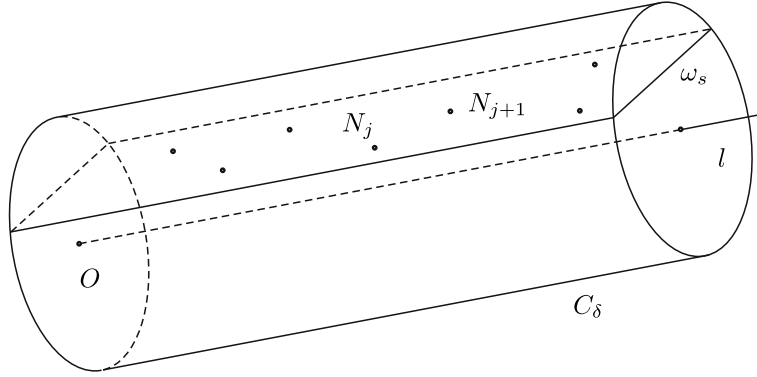


Fig. 3. The intersection of C_δ and ω_s is unbounded.

If the first case, all the distances between N_j and l are bounded from below by some positive constant (which is equal to the distance between ω_s and l). But this is impossible since $d(N_j) \rightarrow 0$ as $j \rightarrow +\infty$.

Further, if the line l lies in the hyperplane ω_s then $\bar{\alpha}$ is the linear combination of the form $\bar{\alpha} = t_1 \bar{u}_1 + \dots + t_s \bar{u}_s$. Denoting the components of \bar{u}_j by $u_{0j}, u_{1j}, \dots, u_{nj}$, we get:

$$\begin{cases} t_1 u_{01} + \dots + t_s u_{0s} = 1, \\ t_1 u_{11} + \dots + t_s u_{1s} = \alpha_1, \\ \dots \\ t_1 u_{n1} + \dots + t_s u_{ns} = \alpha_n. \end{cases} \quad (11)$$

Since $\bar{u}_1, \dots, \bar{u}_s$ are linearly independent then $(n+1) \times s$ -matrix of its components has the maximal rank s . Hence, it contains s linearly independent rows, and let $0 \leq i_1 < i_2 < \dots < i_s \leq n$ be their indices. If it is necessary, we put $\alpha_0 = 1$ and consider the corresponding system of equations extracting from (11), that is

$$\begin{cases} t_1 u_{i_1 1} + \dots + t_s u_{i_1 s} = \alpha_{i_1}, \\ \dots \\ t_1 u_{i_s 1} + \dots + t_s u_{i_s s} = \alpha_{i_s}. \end{cases}$$

Its determinant is nonzero integer. Cramer's formulas implies that the unique solution of this system has the form

$$\begin{cases} t_1 = r_{11} \alpha_{i_1} + \dots + r_{1s} \alpha_{i_s}, \\ \dots \\ t_s = r_{s1} \alpha_{i_1} + \dots + r_{ss} \alpha_{i_s}, \end{cases}$$

where r_{ij} are some rationals. Since $s \leq n$ then there exist at least one equation in (11)

whose index j differs from i_1, \dots, i_s . Thus we get:

$$\begin{aligned}\alpha_j &= t_1 u_{j1} + \dots + t_s u_{js} = \\ &= (r_{11}\alpha_{i_1} + \dots + r_{1s}\alpha_{i_s})u_{j1} + \dots + (r_{s1}\alpha_{i_1} + \dots + r_{ss}\alpha_{i_s})u_{js} = \\ &= q_1\alpha_{i_1} + \dots + q_s\alpha_{i_s},\end{aligned}$$

where $q_1, \dots, q_s \in \mathbb{Q}$. The last relation contradicts to the linear independence of $1, \alpha_1, \dots, \alpha_n$ over the rationals.

This contradiction implies that the hyperplane ω_s does not contain the line l . This proves the lemma.

COROLLARY. *For any vector $\bar{\alpha} = (1, \alpha_1, \dots, \alpha_n)$ whose components are linearly independent over the rationals, for any tuple of real numbers β_1, \dots, β_n and for any ε , $0 < \varepsilon < 0.5$, there exists a constant $c = c(\bar{\alpha}, \varepsilon)$ such that any interval of length c contains at least one value t such that the following inequalities hold: $\|t\alpha_j + \beta_j\| < \varepsilon$, $j = 1, \dots, n$.*

PROOF. We use the notations of Lemma 6. The above arguments imply that the cylinder C with radius $\varepsilon_1 = \varepsilon|\bar{\alpha}|^{-1}$ and axis l passing through the origin in parallel to $\bar{\alpha}$ contains an $(n+1)$ -dimensional parallelepiped Π whose vertices belong to the lattice \mathbb{Z}^{n+1} .

Then the cylinder $C_0 = C + \bar{\beta}$, which is the shift of C to vector $\bar{\beta} = (1, \beta_1, \dots, \beta_n)$, contains a parallelepiped $\Pi_0 = \Pi + \bar{\beta}$. Any shift of Π contains a lattice point. Hence, both Π_0 and any parallelepiped Π_j which is the shift of Π_0 to vector $\bar{\xi}_j = c_0 j \bar{\alpha}$, $j = \pm 1, \pm 2, \dots$, parallel to the axis of the cylinder C_0 , contain the points of the lattice \mathbb{Z}^{n+1} . It is easy to note that the parallelepipeds Π_j have no common points.

Finally, let $M_j = (m_0, \dots, m_n)$ be a lattice point containing in Π_j . The distance between this point and the axis of C_0 does not exceed ε_1 . At the same time, this distance is expressed by (9), where Δ_{ij} is a minor of matrix

$$\begin{pmatrix} 1 & \alpha_1 & \dots & \alpha_n \\ m_0 & m_1 - \beta_1 & \dots & m_n - \beta_n \end{pmatrix}.$$

formed by its columns i, j . Hence, we have

$$|\Delta_{ij}| = |m_0\alpha_j - (m_j - \beta_j)| = |m_0\alpha_j + \beta_j - m_j| < \varepsilon$$

for any j , $1 \leq j \leq n$. By the inequality $\varepsilon < 0.5$, we obtain that $\|m_0\alpha_j + \beta_j\| < \varepsilon$. To end the proof, we note that the first coordinates m_0 of the points M_j form an increasing sequence, whose neighbouring elements differ for at most to $3c_0$.

§3. Large values of the Riemann zeta function on the critical line

In this section, we give a conditional solution of Karatsuba's problem based on the Riemann hypothesis. We also prove a series of statements concerning the existence of large values of the function $S(t)$ on the short segments of the real axis.

THEOREM 1. *Suppose that the Riemann hypothesis is true, and let A be an arbitrary large fixed constant. Then there exist the constants $c_0 = c_0(A) > 0$ and $T_0 = T_0(A)$ such that any interval of the form $(T - H, T + H)$, $H = (1/\pi) \ln \ln \ln T + c$, $T > T_0$, contains at least one point t such that $|\zeta(0.5 + it)| > A$.*

PROOF. Let's fix any positive number $a > 1$ satisfying the condition

$$e^a \sqrt{\frac{\pi}{2a}} \geq \ln A. \quad (12)$$

Extracting real parts in (5), we obtain:

$$\begin{aligned} \int_{-\infty}^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + i(t + u))| du &= \frac{1}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \hat{K}_a\left(\frac{\ln n}{\pi}\right) \cos(t \ln n) - \\ &\quad - 2\pi \int_0^{0.5} \Re K_a(\pi t + \pi i v) dv. \end{aligned} \quad (13)$$

Taking $t = 0$ in (13) and noting that $K_a(\pi i v) = e^{-a \cos \pi v}$, we have:

$$\int_{-\infty}^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + i u)| du = \frac{1}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \hat{K}_a\left(\frac{\ln n}{\pi}\right) - 2\pi \int_0^{0.5} e^{-a \cos(\pi v)} dv. \quad (14)$$

Further, the relation $|K_a(\pi t + \pi i v)| = e^{-a \cosh(\pi t) \cos(\pi v)}$ implies that the last integral in (13) does not exceed

$$2\pi \int_0^{0.5} e^{-a \cosh(\pi t) \cos(\pi v)} dv = 2\pi \int_0^{0.5} e^{-a \cosh(\pi t) \sin(\pi v)} dv < \frac{\pi}{a \cosh(\pi t)} \quad (15)$$

in modulus. Subtracting (14) from (13) and using the estimate (15), we find

$$\begin{aligned} \int_{-\infty}^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + i(t + u))| du - \int_{-\infty}^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + i u)| du &= \\ = 2\pi \int_0^{0.5} e^{-a \cos(\pi v)} dv - \frac{2}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \hat{K}_a\left(\frac{\ln n}{\pi}\right) \sin^2\left(\frac{t}{2} \ln n\right) + \frac{\pi \theta_1}{\cosh(\pi t)}. \end{aligned} \quad (16)$$

Let ε, N be the numbers satisfying the conditions $0 < \varepsilon < 0.5$, $N > N_0 = e^{\pi a \sqrt{2}}$ and depending only on a , whose precise values will be chosen below. Applying Lemmas 1 and 6, we find the constant $c_0 = c_0(a)$ such that any interval of real axis with length c_0 contains at least one point τ such that the inequalities $\|(\tau/(2\pi)) \ln p\| < \varepsilon$ hold true for all primes $p \leq N$. Let us take t to be equal to such value τ from the interval $(T, T + c_0)$ in (16).

Given prime $p \leq N$, we define an integer n_p and real ε_p satisfying the condition $|\varepsilon_p| < \varepsilon$ such that $(t/(2\pi)) \ln p = n_p + \varepsilon_p$. Then we have

$$\sin^2\left(\frac{t}{2} \ln n\right) = \sin^2(\pi k n_p + \pi k \varepsilon_p) = \sin^2(\pi k \varepsilon_p) < (\pi k \varepsilon)^2$$

for any $k \geq 1$ and $n = p^k$.

Let C be the sum in the right-hand side of (16). Denote by C_1 and C_2 the contributions to C from the terms corresponding to $n = p^k$, $k \geq 1$, $p \leq N$ and from all other terms, respectively. Then we have:

$$|C_1| \leq \frac{2}{\pi} (\pi\varepsilon)^2 \sum_{\substack{n=p^k \\ k \geq 1, p \leq N}} \frac{k}{\sqrt{n}} \left| \widehat{K}_a \left(\frac{\ln n}{\pi} \right) \right|.$$

We split the domain of n to the intervals $n \leq N_0$, $N_0 < n \leq N$ and $n > N$ and then denote the corresponding parts of sum by C_3, C_4, C_5 . The estimate $|\widehat{K}_a((1/\pi) \ln n)| \leq \widehat{K}_a(0)$ implies

$$\begin{aligned} |C_3| &\leq 2\pi\varepsilon^2 \widehat{K}_a(0) \sum_{p \leq N_0} \sum_{k=1}^{+\infty} k p^{-k/2} = \\ &= 2\pi\varepsilon^2 \widehat{K}_a(0) \sum_{p \leq N_0} \frac{1}{\sqrt{p}} \left(1 - \frac{1}{\sqrt{p}} \right)^{-2} \leq 2\pi\varepsilon^2 \left(1 - \frac{1}{\sqrt{2}} \right)^{-2} \widehat{K}_a(0) \sum_{p \leq N_0} \frac{1}{\sqrt{p}}. \end{aligned}$$

Let us use the inequality

$$\sum_{p \leq x} \frac{1}{\sqrt{p}} \leq \frac{2.784\sqrt{x}}{\ln x},$$

which is verified for $2 \leq x \leq 1.5 \cdot 10^6$ by Wolfram Mathematica 7.0 and follows from the inequality (3.6) from [23, Th. 2, corollary 1] by Abel's summation formula for $x > 1.5 \cdot 10^6$. Thus we get

$$|C_3| < 45.9\varepsilon^2 \widehat{K}_a(0) \frac{e^{\pi a/\sqrt{2}}}{a} \leq (7\varepsilon)^2 e^{\pi a/\sqrt{2}} \widehat{K}_a(0).$$

Further, the Corollary of Lemma 3 implies

$$\begin{aligned} |C_4| &\leq 2\pi\varepsilon^2 \sum_{\substack{N_0 < n \leq N \\ n = p^k}} k p^{-k/2} \cdot \frac{61.5\sqrt{\pi}}{\sqrt{k \ln p}} \exp \left(-\frac{\pi}{2} \frac{1}{\pi} \ln p^k \right) = \\ &= 123\pi \sqrt{\pi} \varepsilon^2 \sum_{\substack{N_0 < n \leq N \\ n = p^k}} \frac{\sqrt{k}}{p^k \sqrt{\ln p}} \leq 123\pi \sqrt{\pi} \varepsilon^2 \sum_{p \leq N} \frac{1}{\sqrt{\ln p}} \sum_{k=1}^{+\infty} k p^{-k} < \\ &< 123\pi \sqrt{\pi} \varepsilon^2 \sum_{p \leq N} \frac{1}{p \sqrt{\ln p}} \left(1 - \frac{1}{p} \right)^{-2} < 123\pi \sqrt{\pi} \varepsilon^2 \sum_p \frac{p}{(p-1)^2 \sqrt{\ln p}} < 3000\varepsilon^2. \end{aligned}$$

Applying the Corollary of lemma 3 together with the estimate (7) again and noting that $\ln N \geq \pi a \sqrt{2} \geq \pi \sqrt{2}$, we find

$$|C_5| \leq \frac{2}{\pi} \sum_{n > N} \frac{\Lambda_1(n)}{\sqrt{n}} \frac{61.5\sqrt{\pi}}{\sqrt{n \ln n}} = \frac{123}{\sqrt{\pi}} \sum_{n > N} \frac{\Lambda(n)}{n (\ln n)^{3/2}}.$$

Abel's summation formula together with the bound

$$\psi(u) = \sum_{n \leq u} \Lambda(n) \leq c_1 u, \quad c_1 = 1.03883$$

(see [23, Th. 12]), which is valid for any $u > 0$, imply

$$\begin{aligned} \sum_{n > N} \frac{\Lambda(n)}{n(\ln n)^{3/2}} &= - \int_N^{+\infty} (\psi(u) - \psi(N)) d \frac{1}{(\ln u)^{3/2}} \leq -c_1 \int_N^{+\infty} u d \frac{1}{(\ln u)^{3/2}} = \\ &= c_1 \left(\frac{2}{\sqrt{\ln N}} + \frac{1}{(\ln N)^{3/2}} \right). \end{aligned}$$

Since $\ln N \geq \pi a \sqrt{2} \geq \pi \sqrt{2}$, we finally get:

$$\begin{aligned} |C_5| &\leq \frac{123}{\sqrt{\pi}} \frac{2c_1}{\sqrt{\ln N}} \left(1 + \frac{1}{2\pi\sqrt{2}} \right) < \frac{160.5}{\sqrt{\ln N}}, \\ |C_1| &\leq |C_3| + |C_4| + |C_5| < (7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{160.5}{\sqrt{\ln N}}. \end{aligned}$$

Applying the same arguments to the estimation of the sum C_2 , we obtain

$$|C_2| \leq \frac{2}{\pi} \sum_{n > N} \frac{\Lambda_1(n)}{\sqrt{n}} \frac{61.5\sqrt{\pi}}{\sqrt{n \ln n}} < \frac{160.5}{\sqrt{\ln N}}.$$

Thus

$$|C| \leq |C_1| + |C_2| < (7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}},$$

and therefore

$$\begin{aligned} \int_{-\infty}^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + i(t+u))| du &\geq 2 \int_0^{\pi/2} e^{-a \sin v} dv + \\ + 2 \int_0^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + iu)| du &- \left((7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}} + \frac{\pi}{\cosh \pi t} \right). \end{aligned} \tag{17}$$

Now we estimate the modulus of the improper integral in the right-hand side of (17). We split it to the integrals j_1 and j_2 , corresponding to the intervals $0 \leq u \leq 10$ and $u > 10$, respectively. Since the modulus of $\ln |\zeta(0.5 + iu)|$ does not exceed $0.641973 \dots < 2/3 - 1/50$ for $0 \leq u \leq 10$, we find

$$|j_1| < \left(\frac{2}{3} - \frac{1}{50} \right) \int_0^{10} K_a(\pi u) du < \frac{1}{\pi} \left(\frac{1}{3} - \frac{1}{100} \right) \widehat{K}_a(0).$$

Further, the formula for $\widehat{K}_a(0)$ from [24, Ex. 9.1] implies that

$$\frac{7}{8} e^{-a} \sqrt{\frac{2\pi}{a}} < \widehat{K}_a(0) < e^{-a} \sqrt{\frac{2\pi}{a}} \tag{18}$$

for $a > 1$. Hence,

$$\begin{aligned}
|j_2| &\leq \frac{\widehat{K}_a(0)}{\widehat{K}_a(0)} \int_{10}^{+\infty} e^{-a \cosh(\pi u)} |\ln |\zeta(0.5 + iu)|| du \leq \\
&\leq \widehat{K}_a(0) \frac{8}{7} e^a \sqrt{\frac{a}{2\pi}} \int_{10}^{+\infty} \exp(-0.5ae^{\pi u}) |\ln |\zeta(0.5 + iu)|| du = \\
&= \widehat{K}_a(0) \frac{8}{7} e^{-a} \sqrt{\frac{a}{2\pi}} \int_{10}^{+\infty} \exp(-0.5a(e^{\pi u} - 4)) |\ln |\zeta(0.5 + iu)|| du.
\end{aligned}$$

Since $0.5(e^{\pi u} - 4) > 2u^2$ for $u \geq 10$, we find

$$\begin{aligned}
|j_2| &\leq \frac{\widehat{K}_a(0)}{\widehat{K}_a(0)} \int_{10}^{+\infty} e^{-2u^2} |\ln |\zeta(0.5 + iu)|| du \leq \widehat{K}_a(0) \frac{8}{7} e^{-a} \sqrt{\frac{a}{2\pi}} \cdot 1.52 \cdot 10^{-89} < \\
&< 1.5 \cdot 10^{-90} \widehat{K}_a(0).
\end{aligned}$$

Thus we get

$$|j_1| + |j_2| < \frac{1}{\pi} \left(\frac{1}{3} - \frac{1}{100} \right) \widehat{K}_a(0) + 1.5 \cdot 10^{-90} \widehat{K}_a(0) < \frac{\widehat{K}_a(0)}{3\pi}.$$

Obviously we have

$$\int_0^{\pi/2} e^{-a \sin v} dv \geq \int_0^{\pi/2} e^{-av} dv = \frac{1}{a} (1 - e^{-\pi a/2}).$$

Therefore, the inequality (17) implies

$$\begin{aligned}
\int_{-\infty}^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + i(t+u))| du &\geq \frac{2}{a} (1 - e^{-\pi a/2}) - \\
&- \left((7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}} + \frac{\widehat{K}_a(0)}{3\pi} + \frac{\pi}{\cosh \pi t} \right). \quad (19)
\end{aligned}$$

Further, we put $h = (1/\pi)(\ln \ln \ln T - \ln(a/2))$ and split the integral to the sum

$$\left(\int_{-h}^h + \int_h^{+\infty} + \int_{-\infty}^{-h} \right) K_a(\pi u) \ln |\zeta(0.5 + i(t+u))| du = j_3 + j_4 + j_5.$$

The formula

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s \int_1^{+\infty} \frac{\varrho(u)}{u^{s+1}} du,$$

where $\varrho(u) = 0.5 - \{u\}$, $\Re s > 0$, $s \neq 1$ (see [25, Ch. II, Lemma 2]) implies that $0 \leq |\zeta(0.5 + iv)| \leq |v| + 3$ for any real v . Hence,

$$-\infty \leq \ln |\zeta(0.5 + iv)| < \ln(|v| + 3).$$

Passing to the estimate of j_4 , we get:

$$\begin{aligned} -\infty \leq j_4 &= \int_h^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + i(t+u))| du < \int_h^{+\infty} K_a(\pi u) \ln (|t+u|+3) du = \\ &= \left(\int_h^t + \int_t^{+\infty} \right) K_a(\pi u) \ln (|t+u|+3) du = j_6 + j_7. \end{aligned}$$

Estimating the integrals j_6 и j_7 separately, we find

$$\begin{aligned} j_6 &\leq \ln(2t+3) \int_h^{+\infty} \exp(-0.5ae^{\pi u}) du = \frac{1}{\pi} \ln(2t+3) \int_{0.5ae^{\pi h}}^{+\infty} e^{-w} \frac{dw}{w} = \\ &= \frac{1}{\pi} \ln(2t+3) \int_{\ln \ln T}^{+\infty} e^{-w} \frac{dw}{w} < \frac{\ln(2t+3)}{\pi \ln T} \frac{1}{\ln \ln T}. \end{aligned}$$

Similarly,

$$\begin{aligned} j_7 &\leq \int_t^{+\infty} \exp(-0.5ae^{\pi u}) \ln(2u+3) du \leq 2 \int_t^{+\infty} \exp(-0.5ae^{\pi u}) (\ln u) du < \\ &< \frac{2}{\pi} \ln(\pi t/2) e^{-\pi t/2} \exp(-e^{\pi t/2}). \end{aligned}$$

Therefore,

$$-\infty \leq j_4 = j_6 + j_7 < \frac{\ln(2t+3)}{\pi \ln T} \frac{1}{\ln \ln T} + \frac{2}{\pi} \ln(\pi t/2) e^{-\pi t/2} \exp(-e^{\pi t/2}) < \frac{1}{3 \ln \ln T}.$$

The integral j_5 is estimated in the same way. Thus we have:

$$\begin{aligned} j_5 &= \int_h^{+\infty} K_a(\pi u) \ln |\zeta(0.5 + i(t-u))| du < \int_h^{+\infty} K_a(\pi u) \ln (|t-u|+3) du = \\ &= \left(\int_h^{2t} + \int_{2t}^{+\infty} \right) K_a(\pi u) \ln (|t-u|+3) du = j_8 + j_9, \\ j_8 &\leq \ln(t+3) \int_h^{+\infty} K_a(\pi u) du < \frac{\ln(t+3)}{\pi \ln T} \frac{1}{\ln \ln T}, \\ j_9 &< \int_{2t}^{+\infty} K_a(\pi u) \ln(u+3) du < \frac{2}{\pi} \ln(\pi t) e^{-\pi t} \exp(-e^{\pi t}), \end{aligned}$$

and hence $j_5 < (3 \ln \ln T)^{-1}$.

Going back to (19), we obtain

$$\begin{aligned} \int_{-h}^h K_a(\pi u) \ln |\zeta(0.5 + i(t+u))| du &> \frac{2}{a} - \\ &- \left(\frac{2}{a} e^{-\pi a/2} + (7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}} + \frac{2\widehat{K}_a(0)}{3\pi} + \frac{1}{\ln \ln T} \right) > \\ &> \frac{2}{a} - \left(\frac{2}{a} \cdot \frac{1}{4} + (7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}} + \frac{2\widehat{K}_a(0)}{3\pi} \right). \quad (20) \end{aligned}$$

In view of (18), the expression in the brackets does not exceed

$$\begin{aligned} \frac{1}{2a} + (7\varepsilon)^2 \sqrt{\frac{2\pi}{a}} e^{(\pi/\sqrt{2}-1)a} + 3000\varepsilon^2 \frac{321}{\sqrt{\ln N}} + \frac{2}{3\pi} e^{-a} \sqrt{\frac{2\pi}{a}} < \\ < \frac{1}{2a} \left(\frac{3}{2} + 2(7\varepsilon)^2 \sqrt{2\pi a} e^{(\pi/\sqrt{2}-1)a} + 6000a\varepsilon^2 + \frac{642a}{\sqrt{\ln N}} \right). \end{aligned}$$

Now we put

$$\varepsilon = \frac{e^{-2a/3}}{100\sqrt{a}}, \quad N = e^{(3852a)^2}.$$

Then the left-hand side of the last inequality does not exceed

$$\frac{1}{2a} \left(\frac{3}{2} + \frac{\sqrt{2\pi}}{100} e^{-0.1a} + \frac{3}{5} e^{-4a/3} + \frac{1}{6} \right) < \frac{1}{2a} \left(\frac{5}{3} + \frac{1}{6} + \frac{1}{6} \right) = \frac{1}{a}.$$

Now (20) implies that

$$\int_{-h}^h K_a(\pi u) \ln |\zeta(0.5 + i(t+u))| du > \frac{2}{a} - \frac{1}{a} = \frac{1}{a}. \quad (21)$$

Denote by M the maximum of $\ln |\zeta(0.5 + i(t+u))|$ on the segment $|u| \leq h$. Then (21) implies that $M > 0$. Hence, the integral in (21) is less than

$$M \int_{-h}^h K_a(\pi u) du < \frac{M}{\pi} \int_{-\infty}^{+\infty} K_a(u) du = \frac{M}{\pi} \widehat{K}_a(0).$$

Using (18) and (12), we find

$$\frac{M}{\pi} \widehat{K}_a(0) > \frac{1}{a}, \quad M > \frac{\pi}{a} \widehat{K}_a^{-1}(0) > e^a \sqrt{\frac{\pi}{2a}} \geq \ln A.$$

To end the proof, we note that the distance between T and the point u , where the maximum M is attained, does not exceed

$$c_0 + h = \frac{1}{\pi} (\ln \ln \ln T - \ln(a/2)) + c_0.$$

The theorem is proved.

REMARK. In [26], the distribution of the random variable $\sigma(T)$ with the values

$$-2 \ln F(t; 2\pi) + 2 \ln \ln \frac{t}{2\pi} - \frac{3}{2} \ln \ln \ln \frac{t}{2\pi}, \quad t_0 \leq t \leq T,$$

is discussed. In [27], there are some arguments that reinforce the hypothesis that the inequalities

$$\frac{\ln t}{(\ln \ln t)^{2+\varepsilon}} \leq F(t; 2\pi) \leq \frac{\ln t}{(\ln \ln t)^{0.25-\varepsilon}}$$

hold for “almost all” t from the interval $(T, 2T)$, $T \rightarrow +\infty$ and for any $\varepsilon > 0$.

THEOREM 2. *Suppose that the quantity*

$$S_0 = \frac{1}{\pi} \sum_{n=p^{2k+1}} (-1)^k \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) = \frac{1}{\pi} \Im \sum_{n=2}^{+\infty} i^{\Omega(n)} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right)$$

is positive for some $a \geq 1$. Then for any fixed $\varepsilon > 0$ satisfying the condition $0 < \varepsilon < \min(0.5, S_0)$ there exist the constants c_0 and T_0 depending on a and ε only and such that the inequalities

$$\max_{|t-T| \leq H} (\pm S(t)) > \frac{S_0 - \varepsilon}{\pi \widehat{K}_a(0)}$$

hold for any $T \geq T_0$ and $H = (1/\pi) \ln \ln \ln T + c_0$.

PROOF. Extracting the real parts in (5), we obtain:

$$\pi \int_{-\infty}^{+\infty} K_a(\pi u) S(t+u) du = -\frac{1}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \sin(t \ln n) + \frac{\pi \theta_1}{\cosh \pi t}. \quad (22)$$

Let ε_1, N , be the numbers depending on a, ε and such that $0 < \varepsilon_1 < 0.5$, $N \geq e^{\pi a \sqrt{2}}$, whose explicit values will be chosen later. By Lemma 6, there exists a constant $c = c(a, \varepsilon)$ such that any interval of length c contains a point τ such that the inequality

$$\left\| \frac{\tau}{2\pi} \ln p + \frac{1}{4} \right\| < \varepsilon_1 \quad (23)$$

holds for any prime $p \leq N$. Let us take the parameter t in (22) to be equal to such value τ from the interval $(T, T+c)$.

Given prime $p \leq N$, we define an integer n_p and real ε_p satisfying the condition $|\varepsilon_p| < \varepsilon_1$ such that

$$\frac{t}{2\pi} \ln p = n_p + \varepsilon_p - \frac{1}{4}.$$

Then we have

$$\sin(t \ln n) = -\sin\left(\frac{\pi k}{2}\right) \cos(2\pi k \varepsilon_p) + \cos\left(\frac{\pi k}{2}\right) \sin(2\pi k \varepsilon_p)$$

for any $k \geq 1$ and $n = p^k$, $p \leq N$. If k is even then $|\sin(t \ln n)| = |\sin(2\pi k \varepsilon_p)| \leq 2\pi k \varepsilon_1$; otherwise, we have

$$\sin(t \ln n) = (-1)^{(k+1)/2} \cos(2\pi k \varepsilon_p) = (-1)^{(k+1)/2} - 2\theta_2(\pi k \varepsilon_1)^2.$$

Let S be the sum in the right-hand side of (22). Denote by S_1, S_2 and S_3 the contributions to this sum arising from the terms corresponding to the following conditions: $n = p^k$,

$p \leq N$, k is odd; $n = p^k$, $p \leq N$, k is even; $n = p^k$, $p > N$, respectively. Then we have

$$\begin{aligned}
S_1 &= -\frac{1}{\pi} \sum_{\substack{n=p^{2k+1} \\ p \leq N, k \geq 0}} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \left((-1)^{k+1} - 2\theta_2(\pi(2k+1)\varepsilon_1)^2\right) = \\
&= \frac{1}{\pi} \sum_{\substack{n=p^{2k+1} \\ p \leq N, k \geq 0}} (-1)^k \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) + \\
&\quad + 2\theta_3\pi\varepsilon_1^2 \sum_{\substack{n=p^{2k+1} \\ p \leq N, k \geq 0}} (2k+1)^2 \frac{\Lambda_1(n)}{\sqrt{n}} \left|\widehat{K}_a\left(\frac{\ln n}{\pi}\right)\right|. \quad (24)
\end{aligned}$$

Obviously, the last sum in (24) is less than

$$\begin{aligned}
2\pi\varepsilon_1^2 \widehat{K}_a(0) \sum_{p \leq N} \sum_{k=0}^{+\infty} \frac{2k+1}{p^k \sqrt{p}} &= 2\pi\varepsilon_1^2 \widehat{K}_a(0) \sum_{p \leq N} \frac{1}{\sqrt{p}} \left(1 + \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)^{-2} \leq \\
&\leq 12\pi\varepsilon_1^2 \widehat{K}_a(0) \sum_{p \leq N} \frac{1}{\sqrt{p}} < \frac{105\varepsilon_1^2 \widehat{K}_a(0) \sqrt{N}}{\ln N}.
\end{aligned}$$

Further, we replace the interval $p \leq N$ in the first sum in right-hand side of (24) by infinite one. The arising error does not exceed in modulus

$$\frac{1}{\pi} \sum_{n > N} \frac{\Lambda_1(n)}{\sqrt{n}} \left|\widehat{K}_a\left(\frac{\ln n}{\pi}\right)\right| < \frac{81}{\sqrt{\ln N}}. \quad (25)$$

Hence, the difference between S_1 and S_0 is less than

$$\frac{105\varepsilon_1^2 \widehat{K}_a(0) \sqrt{N}}{\ln N} + \frac{81}{\sqrt{\ln N}}.$$

Further,

$$\begin{aligned}
|S_2| &\leq \frac{1}{\pi} \sum_{\substack{n=p^{2k} \\ p \leq N, k \geq 1}} \frac{\Lambda_1(n)}{\sqrt{n}} |\widehat{K}_a(0)| \cdot 4\pi k \varepsilon_1 \leq 2\varepsilon_1 \widehat{K}_a(0) \sum_{p \leq N} \sum_{k=1}^{+\infty} p^{-k} = \\
&= 2\varepsilon_1 \widehat{K}_a(0) \sum_{p \leq N} \frac{1}{p-1} < 3\varepsilon_1 \widehat{K}_a(0) \ln \ln N.
\end{aligned}$$

Obviously, the modulus of S_3 does not exceed the right-hand side of (25).

Therefore, S and S_0 differ by at most

$$\frac{105\varepsilon_1^2 \widehat{K}_a(0) \sqrt{N}}{\ln N} + \frac{162}{\sqrt{\ln N}} + 3\varepsilon_1 \widehat{K}_a(0) \ln \ln N.$$

We put $h = (1/\pi)(\ln \ln \ln T - \ln(a/2))$ and split the improper integral in (22) to the integrals j_1, j_2 and j_3 corresponding to the intervals $|u| \leq h$, $u > h$ and $u < -h$ respectively. If $|v| \geq 280$, the classical Backlund's estimate [28] implies that

$$|S(v)| < 0.1361 \ln |v| + 0.4422 \ln \ln |v| + 4.3451 \leq \left(0.1361 + 0.4422 \frac{\ln \ln 280}{\ln 280} + \frac{4.3451}{\ln 280} \right) \ln |v| < 1.05 \ln |v|. \quad (26)$$

Otherwise, we have the inequality $|S(v)| \leq 1$ (see [29, Tab. 1]). From these estimates, it follows that $|j_2| + |j_3| < 2(\ln \ln T)^{-1}$. Hence,

$$\begin{aligned} \pi \int_{-h}^h K_a(\pi u) S(t+u) du &> \\ &> S_0 - \left(\frac{105\varepsilon_1^2 \widehat{K}_a(0) \sqrt{N}}{\ln N} + \frac{162}{\sqrt{\ln N}} + 3\widehat{K}_a(0)\varepsilon_1 \ln \ln N + \frac{3}{\ln \ln T} \right). \end{aligned} \quad (27)$$

The expression in the brackets is less than

$$\begin{aligned} \frac{105\varepsilon_1^2 \sqrt{N}}{\ln N} e^{-a} \sqrt{\frac{2\pi}{a}} + \frac{162}{\sqrt{\ln N}} + 3e^{-a} \sqrt{\frac{2\pi}{a}} \varepsilon_1 \ln \ln N + \frac{3}{\ln \ln T} < \\ < \frac{(10\varepsilon_1)^2 \sqrt{N}}{\ln N} + \frac{162}{\sqrt{\ln N}} + 3\varepsilon_1 \ln \ln N. \end{aligned} \quad (28)$$

Now we take

$$\varepsilon_1 = \exp \left(- \left(\frac{162}{\varepsilon} \right)^2 \right), \quad N = \exp \left(\left(\frac{324}{\varepsilon} \right)^2 \right).$$

Then the right-hand side of (28) is bounded from above by

$$\left(\frac{\varepsilon}{30} \right)^2 + 6 \exp \left(- \left(\frac{162}{\varepsilon} \right)^2 \right) \ln \left(\frac{324}{\varepsilon} \right) + \frac{\varepsilon}{2} < \varepsilon.$$

Since $0 < \varepsilon < S_0$, the right-hand side of (27) is positive. Denoting $M_1 = \max_{|u| \leq h} S(t+u)$, we have therefore:

$$M_1 > 0, \quad S_0 - \varepsilon < \pi M_1 \int_{-h}^h K_a(\pi u) du < M_1 \widehat{K}_a(0).$$

Thus, $M_1 > (S_0 - \varepsilon) \widehat{K}_a^{-1}(0)$. Since the distance between T and the point $t+u$, where the maximum is attained, is less than $H = h + c = (1/\pi)(\ln \ln \ln T - \ln(a/2)) + c$, the first statement of theorem is proved. The proof of second one is similar. The only difference is that t is chosen now in $(T, T+c)$ to satisfy the inequalities

$$\left\| \frac{t}{2\pi} \ln p - \frac{1}{4} \right\| < \varepsilon_1$$

for all primes $p \leq N$. The theorem is proved.

The very slow convergence of the series S_0 and the absence of the analogue of the identity (14) make the verification of the condition $S_0 > 0$ very difficult. However, a small modification of the above proof allows one to obtain a series of numerical results.

THEOREM 3. *Let a, b, τ be any positive numbers satisfying the conditions $0 < b < \pi/2$, $b\tau > 0.5$, $\gamma = b\tau + 0.5$, $N \geq 2$ be an integer, and let*

$$S_N(u) = \sum_{p \leq N} \arctan \left(\frac{2\sqrt{p}}{p-1} \cos(u\tau \ln p) \right).$$

Further, let

$$\begin{aligned} \kappa &= \kappa(a, b) = 2 \int_0^{+\infty} e^{-a \cos(b) \cosh(u)} du, \\ \zeta_N(\gamma) &= \prod_{p > N} (1 - p^{-\gamma})^{-1} = \zeta(\gamma) \prod_{p \leq N} (1 - p^{-\gamma}), \end{aligned}$$

and let

$$I = \frac{1}{\pi} \left(\int_0^{+\infty} K_a(u) S_N(u) du - \kappa \ln \zeta_N(\gamma) \right) > 0.$$

Then, for any fixed ε , $0 < \varepsilon < \varepsilon_0(a, b, \tau)$, there exists a constant $c_0 = c_0(\varepsilon; a, b, \tau)$ such that the inequalities

$$\max_{|T-t| \leq H} (\pm S(t)) > \frac{I - \varepsilon}{\widehat{K}_a(0)}$$

hold for any $T \geq T_0(\varepsilon; a, b, \tau)$ and $H = \tau \ln \ln \ln T + c_0$.

PROOF. Setting $f(u) = (1/\tau)K_a(u/\tau)$ in Lemma 4 and extracting imaginary parts, we get

$$\frac{1}{\tau} \int_{-\infty}^{+\infty} K_a\left(\frac{u}{\tau}\right) S(t+u) du = C + \frac{\pi\theta_1}{a \cosh(t/\tau)}, \quad (29)$$

where

$$C = -\frac{1}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a(\tau \ln n) \sin(\tau \ln n).$$

We denote

$$c = \kappa\left(\frac{4}{3} + 7 \ln \zeta(\gamma)\right), \quad \varepsilon_0 = \min\left(0.5, \frac{I}{c}, \frac{\varepsilon}{c}\right)$$

and take an arbitrary fixed numbers ε_1 and N satisfying the conditions $0 < \varepsilon_1 < \varepsilon_0$, $N > 2$. By Corollary of Lemma 6, there exists a constant c_0 depending only on ε_1 , N and such that any interval of length c_0 contains a point τ such that the inequality (23) holds for any prime $p \leq N$. Suppose that t is such value from the interval $(T, T + c_0)$. Similarly to the proof of Theorem 2, we split the sum C to the sums C_1, C_2 and C_3 . Thus

we get $C_1 = C_0 + \theta_2 C_4$, where

$$C_0 = \frac{1}{\pi} \sum_{\substack{n=p^{2k+1}, k \geq 0 \\ p \leq N}} (-1)^k \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a(\tau \ln n),$$

$$C_4 = 2\pi\varepsilon_1^2 \sum_{\substack{n=p^{2k+1}, k \geq 0 \\ p \leq N}} (2k+1)^2 \frac{\Lambda_1(n)}{\sqrt{n}} |\widehat{K}_a(\tau \ln n)|.$$

Moreover,

$$|C_2| \leq 4\varepsilon_1 \sum_{\substack{n=p^{2k}, k \geq 1 \\ p \leq N}} k \frac{\Lambda_1(n)}{\sqrt{n}} |\widehat{K}_a(\tau \ln n)|, \quad |C_3| \leq \frac{1}{\pi} \sum_{n=p^k, p > N} \frac{\Lambda_1(n)}{\sqrt{n}} |\widehat{K}_a(\tau \ln n)|.$$

The application of Lemma 2 yields:

$$\begin{aligned} |C_2| &\leq 4\varepsilon_1 \sum_{\substack{n=p^{2k}, k \geq 1 \\ p \leq N}} \frac{k}{2k\sqrt{n}} \kappa e^{-b\tau \ln n} = 2\kappa\varepsilon_1 \sum_{p \leq N} \sum_{k=1}^{+\infty} p^{-2k\gamma} = \\ &= 2\kappa\varepsilon_1 \sum_{p \leq N} p^{-2\gamma} (1 - p^{-2\gamma})^{-1} \leq \frac{2\kappa\varepsilon_1}{1 - 2^{-2\gamma}} \ln \zeta(2\gamma) < \frac{2\kappa\varepsilon_1}{1 - 2^{-2}} \ln \zeta(2) < \frac{4}{3} \kappa\varepsilon_1, \\ |C_3| &\leq \frac{\kappa}{\pi} \sum_{\substack{n=p^k, k \geq 1 \\ p > N}} \frac{\Lambda_1(n)}{n^\gamma} = \frac{\kappa}{\pi} \sum_{p > N} \ln (1 - p^{-\gamma})^{-1} = \frac{\kappa}{\pi} \ln \left(\zeta(\gamma) \prod_{p \leq N} (1 - p^{-\gamma}) \right) = \\ &= \frac{\kappa}{\pi} \ln \zeta_N(\gamma), \end{aligned}$$

and finally

$$\begin{aligned} |C_4| &\leq 2\pi\kappa\varepsilon_1^2 \sum_{\substack{n=p^{2k+1}, k \geq 0 \\ p \leq N}} \frac{2k+1}{n^\gamma} = 2\pi\kappa\varepsilon_1^2 \sum_{p \leq N} \sum_{k=0}^{+\infty} \frac{2k+1}{p^{(2k+1)\gamma}} = \\ &= 2\pi\kappa\varepsilon_1^2 \sum_{p \leq N} \frac{1}{p^\gamma} \frac{1 + p^{-2\gamma}}{(1 - p^{-2\gamma})^2} \leq 2\pi\kappa\varepsilon_1^2 \frac{1 + 2^{-2\gamma}}{(1 - 2^{-2\gamma})^2} \sum_{p \leq N} \frac{1}{p^\gamma} < \frac{40\pi}{9} \kappa\varepsilon_1^2 \ln \zeta(\gamma). \end{aligned}$$

Transforming the sum C_0 , we obtain

$$\begin{aligned}
C_0 &= \frac{1}{\pi} \sum_{\substack{n=p^{2k+1}, k \geq 0 \\ p \leq N}} (-1)^k \frac{\Lambda_1(n)}{\sqrt{n}} \int_{-\infty}^{+\infty} K_a(u) e^{-iu\tau \ln n} du = \\
&= \frac{1}{\pi} \int_{-\infty}^{+\infty} K_a(u) \left(\sum_{\substack{n=p^{2k+1}, k \geq 0 \\ p \leq N}} (-1)^k \frac{\Lambda_1(n)}{\sqrt{n}} n^{-iu\tau} \right) du = \\
&= \frac{1}{\pi} \int_{-\infty}^{+\infty} K_a(u) \sum_{p \leq N} \left(\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} \left(\frac{p^{-iu\tau}}{\sqrt{p}} \right)^{2k+1} \right) du = \\
&= \frac{1}{2\pi i} \int_0^{+\infty} K_a(u) \sum_{p \leq N} \left\{ \ln \left(1 + \frac{ip^{-iu\tau}}{\sqrt{p}} \right) - \ln \left(1 - \frac{ip^{-iu\tau}}{\sqrt{p}} \right) + \right. \\
&\quad \left. + \ln \left(1 + \frac{ip^{iu\tau}}{\sqrt{p}} \right) - \ln \left(1 - \frac{ip^{iu\tau}}{\sqrt{p}} \right) \right\} du. \quad (30)
\end{aligned}$$

For fixed $p \leq N$, we denote

$$z_1 = 1 + \frac{ip^{iu\tau}}{\sqrt{p}} = |z_1|e^{i\varphi_1}, \quad z_2 = 1 - \frac{ip^{iu\tau}}{\sqrt{p}} = |z_2|e^{i\varphi_2},$$

where $-\pi < \varphi_1, \varphi_2 \leq \pi$. Then the summands in (30) take the form

$$\ln \bar{z}_2 - \ln \bar{z}_1 + \ln z_1 - \ln z_2 = \ln \frac{\bar{z}_2}{z_2} - \ln \frac{\bar{z}_1}{z_1} = 2i(\varphi_1 - \varphi_2).$$

Writing $\alpha_p = u\tau \ln p$ and noting that

$$z_1 = 1 - \frac{\sin \alpha_p}{\sqrt{p}} + \frac{i \cos \alpha_p}{\sqrt{p}},$$

we find

$$\tan \varphi_1 = \tan(\arg z_1) = \frac{(\cos \alpha_p)/\sqrt{p}}{1 - (\sin \alpha_p)/\sqrt{p}} = \frac{\cos \alpha_p}{\sqrt{p} - \sin \alpha_p}.$$

Similarly,

$$\tan \varphi_2 = \tan(\arg z_2) = -\frac{\cos \alpha_p}{\sqrt{p} + \sin \alpha_p}.$$

Hence

$$\tan(\varphi_1 - \varphi_2) = \frac{\tan \varphi_1 - \tan \varphi_2}{1 + \tan \varphi_1 \tan \varphi_2} = \frac{2\sqrt{p}}{p-1} \cos \alpha_p,$$

and therefore

$$\begin{aligned}
\varphi_1 - \varphi_2 &= \arctan \left(\frac{2\sqrt{p}}{p-1} \cos \alpha_p \right), \\
C_0 &= \frac{1}{\pi} \int_0^{+\infty} K_a(u) \sum_{p \leq N} \arctan \left(\frac{2\sqrt{p}}{p-1} \cos \alpha_p \right) du = \frac{1}{\pi} \int_0^{+\infty} K_a(u) S_N(u) du.
\end{aligned}$$

Summing the above bounds, we conclude that the difference between C_0 and the right-hand side of (29) does not exceed in modulus

$$\begin{aligned} \frac{\kappa}{\pi} \ln \zeta_N(\gamma) + \frac{4}{3} \kappa \varepsilon_1 + \frac{40\pi}{9} \kappa \varepsilon_1^2 \ln \zeta(\gamma) + \frac{\pi}{a \cosh(t/\tau)} < \\ < \frac{\kappa}{\pi} \ln \zeta_N(\gamma) + \kappa \varepsilon_1 \left(\frac{4}{3} + 7 \ln \zeta(\gamma) \right) - \frac{3}{\ln \ln T} = \frac{\kappa}{\pi} \ln \zeta_N(\gamma) + c \varepsilon_1 - \frac{3}{\ln \ln T}. \end{aligned}$$

Let $h = \tau(\ln \ln \ln T - \ln(a/2))$. Splitting the integral in (29) to the sum

$$j_1 + j_2 + j_3 = \frac{1}{\tau} \left(\int_{-h}^h + \int_h^{+\infty} + \int_{-\infty}^{-h} \right) K_a\left(\frac{u}{\tau}\right) S(t+u) du$$

and using the same bounds for $S(u)$ as in the proof of Theorem 2, we find: $|j_2| + |j_3| < 3(\ln \ln T)^{-1}$. Hence,

$$j_1 = \frac{1}{\tau} \int_{-h}^h K_a\left(\frac{u}{\tau}\right) S(t+u) du > C_0 - \frac{\kappa}{\pi} \ln \zeta_N(\gamma) - c \varepsilon_1 = I - c \varepsilon_1.$$

Since $0 < \varepsilon_1 < I/c$, the right-hand side of the last inequality is strictly positive, and so is the quantity $M_1 = \max_{|u| \leq h} S(t+u)$. Obviously, we have $j_1 < M_1 \hat{K}_a(0)$, and therefore

$M_1 > (I - \varepsilon)/\hat{K}_a(0)$. The lower bound of $M_2 = \max_{|u| \leq h} (-S(t+u))$ is established by similar arguments. The theorem is proved.

The condition $I > 0$ can be checked without significant difficulties. Let

$$\mu = \frac{I}{\hat{K}_a(0)} = \frac{1}{\pi \hat{K}_a(0)} \left(\int_0^{+\infty} K_a(u) S_N(u) du - \kappa \ln \zeta_N(\gamma) \right).$$

Taking $a = 3$, $b = 7/5$, $\tau = 2/5$ and choosing $N = p_n$ from the table below, we find that

n	μ
16 500	1.005 075 13 ...
78 000	2.006 322 98 ...
2 500 000	3.001 263 70 ...

COROLLARY. *If the Riemann hypothesis is true, then there exist the constants c_0 and T_0 such that the inequalities*

$$\max_{|t-T| \leq H} (\pm S(t)) > 3 + 10^{-3}$$

hold for any $T \geq T_0$ and $H = 0.4 \ln \ln \ln T + c_0$.

THEOREM 4. *Suppose that the Riemann hypothesis is true. Then for an arbitrary large fixed $A \geq 1$, there exist constants T_0, c_0 and h depending only on A and such that the inequality*

$$\min_{|t-T| \leq H} (S(t+h) - S(t-h)) < -A$$

holds with $T \geq T_0$ and $H = (1/\pi) \ln \ln \ln T + c_0$.

PROOF. Let $a > 1$ and $0 < h < 1$ be fixed numbers. Replacing t in (5) by $t + h$ and $t - h$ and subtracting the corresponding relations, we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} K_a(\pi u) (\ln \zeta(0.5 + i(t + h)) - \ln \zeta(0.5 + i(t - h))) du = \\ = \frac{2}{\pi i} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \sin(h \ln n) n^{-it} - \\ - 2\pi \int_0^{0.5} (K_a(\pi(t + h + iv)) - K_a(\pi(t - h + iv))) dv. \end{aligned} \quad (31)$$

Taking imaginary parts in (31), we get

$$\begin{aligned} \pi \int_{-\infty}^{+\infty} K_a(\pi u) (S(t + h + u) - S(t - h + u)) du = \\ = -\frac{2}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \sin(h \ln n) \cos(t \ln n) - \\ - 2\pi \Im \int_0^{0.5} (K_a(\pi(t + h + iv)) - K_a(\pi(t - h + iv))) dv. \end{aligned} \quad (32)$$

If $t = 0$ then the integral in the right-hand side in (32) has the form

$$\begin{aligned} -2\pi \Im \int_0^{0.5} e^{-a \cosh(\pi h) \cos(\pi v)} (e^{-ia \sinh(\pi h) \sin(\pi v)} - e^{ia \sinh(\pi h) \sin(\pi v)}) dv = \\ = 4\pi \int_0^{0.5} e^{-a \cosh(\pi h) \cos(\pi v)} \sin(a \sinh(\pi h) \sin(\pi v)) dv. \end{aligned}$$

Hence, we have

$$\begin{aligned} -\frac{2}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \sin(h \ln n) = \\ = -4\pi \int_0^{0.5} e^{-a \cosh(\pi h) \cos(\pi v)} \sin(a \sinh(\pi h) \sin(\pi v)) dv + \\ + \pi \int_{-\infty}^{+\infty} K_a(\pi u) (S(u + h) - S(u - h)) du. \end{aligned} \quad (33)$$

Let ε, N be the numbers satisfying the conditions $0 < \varepsilon < 0.5$, $N > e^{\pi a \sqrt{2}}$ and depending only on a , whose precise values will be chosen below.

By Lemma 6, given ε, N satisfying the conditions $0 < \varepsilon < 0.5$, $N > e^{\pi a \sqrt{2}}$, there exists a constant c such that any interval of length c contains a point τ such that $\|(\tau/(2\pi)) \ln p\| < \varepsilon$ for any prime $p \leq N$. Taking t in (32) to be equal to such value from the interval $(T, T + c)$, estimating the integral in the right-hand side of (32) by

$2\pi(a \cosh \pi(t-h))^{-1}$ and using the identity (33), we transform the right-hand side of (32) to the form

$$\begin{aligned} & -4\pi \int_0^{0.5} e^{-a \cosh(\pi h) \cos(\pi v)} \sin(a \sinh(\pi h) \sin(\pi v)) dv + \\ & \quad + \pi \int_{-\infty}^{+\infty} K_a(\pi u) (S(u+h) - S(u-h)) du + \\ & \quad + \frac{4}{\pi} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \sin(h \ln n) \sin^2\left(\frac{t}{2} \ln n\right) + \frac{2\pi\theta_1}{a \cosh \pi(t-h)}. \end{aligned} \quad (34)$$

The sum over n in the right-hand side of (34) is estimated in the same way as the sum C in Theorem 1 and does not exceed

$$2\left((7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}}\right)$$

in modulus. In view of (26), the improper integral in (33) does not exceed

$$\begin{aligned} 2\pi \int_{-279}^{279} K_a(\pi u) du + \pi \left(\int_{279}^{+\infty} + \int_{-\infty}^{-279} \right) K_a(\pi u) \cdot 2.1 \ln(|u|+1) du < \\ < 2\widehat{K}_a(0) + 10^{-100} \widehat{K}_a(0) < 2.1\widehat{K}_a(0) \end{aligned}$$

in absolute value. Hence, changing the signs in (34), we get

$$\begin{aligned} & \pi \int_{-\infty}^{+\infty} K_a(\pi u) (S(u+h) - S(u-h)) du > \\ & > 4\pi \int_0^{0.5} e^{-a \cosh(\pi h) \cos(\pi v)} \sin(a \sinh(\pi h) \sin(\pi v)) dv - \\ & - 2\left((7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{3210}{\sqrt{\ln N}} + 2.1\widehat{K}_a(0) + \frac{2\pi}{a \cosh \pi(t-h)}\right). \end{aligned} \quad (35)$$

Now we take $h = (2\pi a)^{-1}$ and estimate the integral in the right-hand side of (35) from below. Since

$$\sin(a \sinh(\pi h) \sin(\pi v)) \geq \sin\left(a\pi h \cdot \frac{2}{\pi} \pi v\right) = \sin v \geq \frac{2}{\pi} v, \quad \cosh \pi h < \cosh \frac{1}{2} < \frac{8}{7},$$

the integral under considering is greater than

$$\begin{aligned} 4\pi \int_0^{0.5} e^{-(8a/7) \cos(\pi v)} \frac{2}{\pi} v dv &= \frac{8}{\pi^2} \int_0^{\pi/2} e^{-(8a/7) \cos w} w dw = \\ &= \frac{8}{\pi^2} \int_0^{\pi/2} e^{-(8a/7) \sin w} \left(\frac{\pi}{2} - w\right) dw \geq \frac{2}{\pi} \int_0^{\pi/4} e^{-(8a/7) \sin w} dw \geq \\ &\geq \frac{2}{\pi} \int_0^{\pi/4} e^{-(8a/7) w} dw = \frac{7}{4\pi a} (1 - e^{-2\pi a/7}) > \frac{7}{4\pi a} (1 - e^{-2\pi/7}) > \frac{0.33}{a}. \end{aligned}$$

Therefore,

$$\pi \int_{-\infty}^{+\infty} K_a(\pi u) (S(t+u-h) - S(t+u+h)) du > \frac{0.33}{a} - \left(2(7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{642}{\sqrt{\ln N}} + 2.1\widehat{K}_a(0) + \frac{2\pi}{a \cosh \pi(t-h)} \right).$$

Let $H_0 = (1/\pi)(\ln \ln \ln T - \ln(a/2))$. Then the sum of integrals over the intervals $(-\infty, -H_0)$ and $(H_0, +\infty)$ in the right-hand side is less than $(\ln \ln T)^{-1}$ in modulus. Thus we get

$$\pi \int_{-H_0}^{H_0} K_a(\pi u) (S(t+u-h) - S(t+u+h)) du > \frac{0.33}{a} - \left(2(7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 6000\varepsilon^2 + \frac{642}{\sqrt{\ln N}} + 2.1\widehat{K}_a(0) + \frac{2}{\ln \ln T} \right). \quad (36)$$

Suppose now that $a > 8$ and take $\varepsilon = e^{-2a/3}/(65\sqrt{a})$, $N = e^{(c_1 a)^2}$, $c_1 = 2^{16}$. Then

$$\begin{aligned} (2(7\varepsilon)^2 \widehat{K}_a(0) e^{\pi a/\sqrt{2}} + 6000\varepsilon^2) &< 98\varepsilon^2 e^{-a} \sqrt{\frac{2\pi}{a}} e^{\pi a/\sqrt{2}} + 6000\varepsilon^2 < \\ &< \frac{98\sqrt{2\pi}}{65^2} \frac{e^{-0.1a}}{\sqrt{a}} \frac{1}{a} + \frac{6000}{65^2} \frac{e^{-4a/3}}{a} < \frac{10^{-2}}{a}, \\ \frac{642}{\sqrt{\ln N}} &= \frac{642}{2^{16}a} < \frac{10^{-2}}{a}, \quad 2.1\widehat{K}_a(0) + \frac{2}{\ln \ln T} < 2.1e^{-a} \sqrt{2\pi a} \frac{1}{a} < \frac{5 \cdot 10^{-3}}{a}. \end{aligned}$$

Thus, the right-hand side of (36) is bounded from below by

$$\frac{0.33}{a} - \left(\frac{2 \cdot 10^{-2}}{a} + \frac{5 \cdot 10^{-3}}{a} \right) > \frac{0.3}{a}.$$

Hence, the value

$$M_0 = \max_{|u| \leq H_0} (S(t+u-h) - S(t+u+h))$$

is positive, and the left-hand side of (36) does not exceed $M_0 \widehat{K}_a(0)$. Therefore,

$$M_0 > \frac{3\widehat{K}_a^{-1}(0)}{10a} > \frac{3e^a}{10\sqrt{2\pi a}} > \frac{e^a}{10\sqrt{a}}.$$

Choosing $a > 8$ such that

$$\frac{e^a}{10\sqrt{a}} > A,$$

we arrive at the assertion of the theorem. The theorem is proved.

In [30], [9], [14], [4] и [31], one can find some other examples of application the function $K_a(z)$ to the theory of $\zeta(s)$.

The key ingredient of the proof of the unboundedness of $|\zeta(0.5 + it)|$ on the segment $|t - T| \ll \ln \ln \ln T$ is the presence of the term

$$2\pi \int_0^{0.5} e^{-a \cos(\pi v)} dv$$

in the right-hand side of (14). It follows from the proof of (4) that the pole of $\zeta(s)$ at the point $s = 1$ is the reason of the appearance of that term. In view of this, it is interesting to prove the analogue of Theorem 1 for the functions that are “similar” to $\zeta(s)$ but have no pole at the point $s = 1$ (for example, for Dirichlet’s L -function $L(s, \chi_4)$, where χ_4 is non-principal character mod 4).

§4. The distribution of zeros of zeta-function.

The above theorems allow one to establish some new statements concerning the distribution of zeros of the Riemann zeta function. Here we also suppose that the Riemann hypothesis is true.

Let $N(t)$ be the number of zeros of $\zeta(s)$ whose ordinate is positive and does not exceed t . Then it is known that

$$N(t) = \frac{1}{\pi} \vartheta(t) + 1 + S(t) = \frac{t}{2\pi} \ln \frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O(t^{-1}),$$

where $\vartheta(t)$ denotes the increment of a continuous branch of the argument of the function $\pi^{-s/2} \Gamma(s/2)$ along the line segment joining the points $s = 0.5$ and $s = 0.5 + it$. Then the Gram’s point t_n ($n \geq 0$) is defined as a unique solution of the equation $\vartheta(t_n) = (n - 1)\pi$ with the condition $\vartheta'(t_n) > 0$. It is easy to check that the number of zeros of $\zeta(0.5 + it)$ lying in the Gram’s interval $G_n = (t_{n-1}, t_n]$ is equal to

$$N(t_n + 0) - N(t_{n-1} + 0) = 1 + \Delta(n) - \Delta(n - 1), \quad (37)$$

where $\Delta(n) = S(t_n + 0)$. Since the segment $[0, T]$ contains

$$\frac{1}{\pi} \vartheta(T) + O(1) = N(T) + O(\ln T)$$

Gram’s intervals G_n , there is precisely one zero of $\zeta(0.5 + it)$ per one Gram’s interval G_n “in the mean”. That is the reason why the difference $\Delta(n) - \Delta(n - 1)$ in (37) is the deviation of number of zeros of $\zeta(0.5 + it)$ in the interval G_n from its mean value, that is, 1.

In 1946, A. Selberg [18] proved that the interval G_n contains no zeros of $\zeta(0.5 + it)$ for positive proportion of n , and contains at least two zeros for positive proportion of n at the same time. These facts show the evident irregularity in the distribution of zeta zeros.

However, nothing is known about the distribution of Gram’s intervals G_n which are “free” of zeros of $\zeta(0.5 + it)$. The below theorem establishes an upper bound for the

length $h = h(t)$ of the interval $(t, t + h)$ which certainly contains an “empty” Gram’s interval G_n .

THEOREM 5. *Suppose that the Riemann hypothesis is true and let ε be any fixed positive constant. Then there exist constants $T_0 = T_0(\varepsilon)$ and $c_0 = c_0(\varepsilon)$ such that any segment $[T - H, T + H]$, where $T \geq T_0$ and $H = (1/\pi) \ln \ln \ln T + c_0$, contains at least $N = [0.1\sqrt{\varepsilon} \exp((\pi\varepsilon)^{-1})]$ Gram’s intervals $G_n = (t_{n-1}, t_n]$ that do not contain zeros of $\zeta(0.5 + it)$. Moreover, there exist at least N intervals among the above “empty” Gram’s intervals that lie in the same segment of length ε .*

PROOF. Let $a = (\pi\varepsilon)^{-1}$, $h = (2\pi a)^{-1} = 0.5\varepsilon$ and suppose ε to be so small that $M = e^a/(10\sqrt{a}) \geq 5$. By Theorem 4, there exist constants $T_0 = T_0(\varepsilon)$ and $c_1 = c_1(\varepsilon)$ such that the inequality

$$\min_{|t-T| \leq H} (S(t+h) - S(t-h)) \leq -M$$

holds for any $T \geq T_0$ with $H = (1/\pi) \ln \ln \ln T + c_1$.

Let k be sufficiently large and suppose that $t_{k-1} \leq a < b \leq t_k$. If $S(t)$ has no discontinuities at (a, b) , then the Riemann-von Mangoldt formula together with Lagrange’s mean value theorem imply that

$$\begin{aligned} S(b) - S(a) &= (b-a)S'(c) = (b-a) \left(-\frac{1}{2\pi} \ln \frac{c}{2\pi} + o(1) \right) = \\ &= -(b-a)(L_k + o(1)), \quad L_k = \frac{1}{2\pi} \ln \frac{t_k}{2\pi} \end{aligned} \quad (38)$$

for some c , $a < c < b$. The relation (38) holds true if a or b coincides with the ordinates of zeta zeros. In this cases, one should replace $S(a)$, $S(b)$ by $S(a+0)$, $S(b-0)$, respectively.

Suppose that $\gamma_{(1)} < \dots < \gamma_{(r)}$ are all the ordinates of zeros of $\zeta(s)$ lying on $[a, b]$, and let $\kappa_{(1)}, \dots, \kappa_{(k)}$ be their multiplicities. Then we have:

$$\begin{aligned} S(b-0) - S(a+0) &= (S(b-0) - S(\gamma_{(k)}+0)) + (S(\gamma_{(k)}+0) - S(\gamma_{(k)}-0)) + \\ &+ (S(\gamma_{(k)}-0) - S(\gamma_{(k-1)}+0)) + \dots + (S(\gamma_{(1)}+0) - S(\gamma_{(1)}-0)) + (S(\gamma_{(1)}-0) - S(a+0)) \\ &= \kappa_{(1)} + \dots + \kappa_{(k)} - (b-a)(L_k + o(1)) \geq -(b-a)(L_k + o(1)) \geq \\ &\geq -(t_k - t_{k-1})(L_k + o(1)) = -1 - o(1) \end{aligned} \quad (39)$$

(see Fig. 4).

Now we define m and n from the relations $t_{m-1} < \tau - h \leq t_m$, $t_n \leq \tau + h < t_{n+1}$. Suppose first that both points $\tau \pm h$ differs from the ordinates of zeta zeros. By (39), we have:

$$S(t_m - 0) - S(\tau - h) \geq -1 - o(1), \quad S(\tau + h) - S(t_n + 0) \geq -1 - o(1),$$

and hence

$$\Delta(m) = S(t_m + 0) \geq S(t_m - 0) \geq S(\tau - h) - 1 - o(1), \quad (40)$$

$$\Delta(n) = S(t_n + 0) \leq S(\tau + h) + 1 + o(1). \quad (41)$$

Subtracting (40) from (41), we find:

$$\Delta(n) - \Delta(m) \leq M + 2 + o(1) < M + 3.$$

Suppose now that $\tau + h$ is the ordinate of multiplicity $\kappa \geq 1$. Then (39) implies

$$S(\tau + h - 0) - S(t_{n-1} + 0) \geq -2 - o(1),$$

and therefore

$$\Delta(n-1) \leq S(\tau + h) + 2 + o(1) = S(\tau + h) - 0.5\kappa + 2 + o(1) \leq S(\tau + h) + 1.5 + o(1). \quad (42)$$

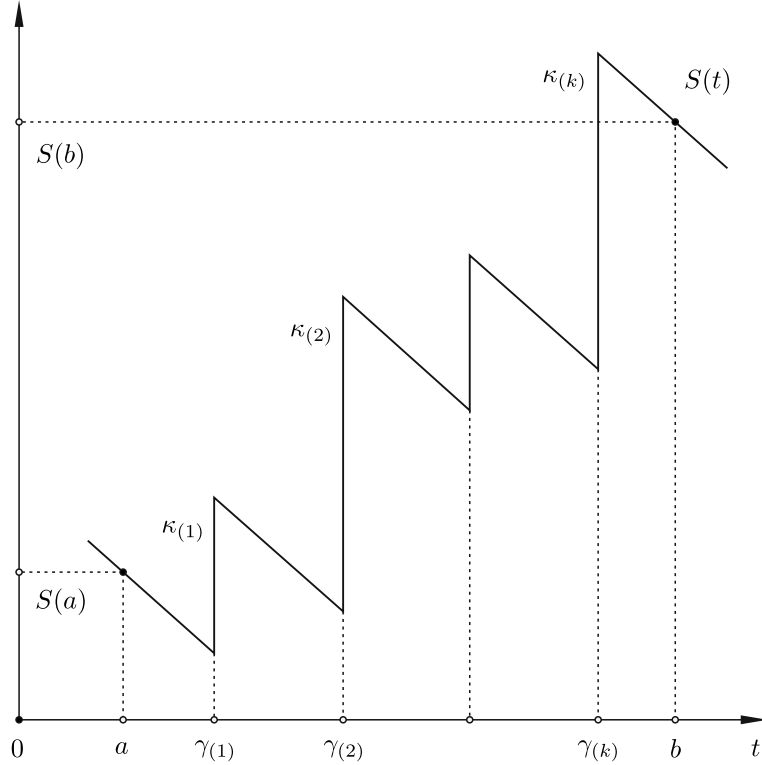


Fig. 4. At each point $\gamma_{(r)}$ of discontinuity, the function $S(t)$ makes a jump equal to the multiplicity of the ordinate $\gamma_{(r)}$, that is, to the sum of multiplicities of all zeta zeros with this point as ordinate.

In view of (40), we get

$$\Delta(n-1) - \Delta(m) \leq M + 2.5 + o(1) < M + 3.$$

Similarly, if $\tau - h$ is an ordinate of a zero of $\zeta(s)$, then

$$S(t_{m+1} - 0) - S(\tau - h) \geq -2 - o(1),$$

and hence

$$\Delta(m+1) = S(t_{m+1} + 0) \geq S(t_{m+1} - 0) \geq S(\tau + h + 0) - 2 - o(1) \geq S(\tau - h) - 1.5 - o(1). \quad (43)$$

Taking (41) into account, we find

$$\Delta(n) - \Delta(m+1) \leq M + 2.5 + o(1) < M + 3.$$

Finally, let both the points $\tau \pm h$ be the ordinates. By (42) and (43), we then have

$$\Delta(n-1) - \Delta(m+1) \leq M + 3 + o(1) < M + 3 + 10^{-4}.$$

The above estimates imply that the smallest difference among $\Delta(n-i) - \Delta(m+j)$, $0 \leq i, j \leq 1$, does not exceed $M + 3 + 10^{-4}$ in any case. Denote by n_1 and m_1 the corresponding values of $n-i$ and $m+j$ and set $N = \lfloor -(M + 3 + 10^{-4}) \rfloor$. Since $N \geq 1$, we get

$$(\Delta(n_1) - \Delta(n_1-1)) + (\Delta(n_1-1) - \Delta(n_1-2)) + \dots + (\Delta(m_1+1) - \Delta(m_1)) \leq -N. \quad (44)$$

Formula (37) implies that $\Delta(k) - \Delta(k-1) \geq -1$ and the equality takes place if and only if Gram's interval G_k is free of zeros of $\zeta(0.5 + it)$. Thus, (44) means that there are at least N negative differences (i.e. equal to -1) among $\Delta(k) - \Delta(k-1)$, $k = m+1, \dots, n$. Hence, there are at least N intervals among the intervals G_k , $k = m+1, \dots, n$, which are free of zeros of $\zeta(0.5 + it)$.

To end the proof, we note that

$$N \geq \frac{e^a}{10\sqrt{a}} - 4 > \frac{e^a}{16\sqrt{a}} = \frac{\sqrt{\pi\varepsilon}}{16} \exp((\pi\varepsilon)^{-1}) > 0.1\sqrt{\varepsilon} \exp((\pi\varepsilon)^{-1}),$$

and that all the intervals G_k , $k = m+1, \dots, n$ are contained in the segment $[\tau-h, \tau+h]$ of length $2h = \varepsilon$. Theorem is proved.

The Corollary of Theorem 3 implies similar (but weaker) result for the distribution of intervals G_n containing at least two zeros of $\zeta(s)$.

THEOREM 6. *Suppose that the Riemann hypothesis is true. Then there exist constants $T_0 = T_0(\varepsilon)$ and $c_0 = c_0(\varepsilon)$ such that any segment $[T-H, T+H]$, where $T \geq T_0$ and $H = 0.8 \ln \ln \ln T + c_0$, contains an interval G_k with at least two zeros of $\zeta(s)$.*

PROOF. By Corollary of Theorem 3, for sufficiently large c and $H_1 = 0.4 \ln \ln \ln T_1 + c$, the interval $(T_1 - H_1, T_1 + H_1)$ contains a point τ_1 such that $S(\tau_1) < -3 - 10^{-3}$, and the interval $(T_1 + H_1, T_1 + 3H_1)$ contains a point τ_2 such that $S(\tau_2) > 3 + 10^{-3}$.

We define m, n by the inequalities $t_m < \tau_1 \leq t_{m+1}$, $t_{n-1} < \tau_2 \leq t_n$. Using the same arguments as in the proof of Theorem 4 together with the inequalities $\tau_1 < \tau_2$, $S(\tau_2) - S(\tau_1) > 6 + 2 \cdot 10^{-3}$, we find

$$S(\tau_1 - 0) - S(t_m + 0) \geq -1 - o(1),$$

and hence

$$-\Delta(m) \geq -S(\tau_1 - 0) - 1 - o(1) \geq -S(\tau_1) - 1 - o(1).$$

Similarly,

$$S(t_n + 0) - S(\tau_2) = (S(t_n + 0) - S(t_n - 0)) + (S(t_n - 0) - S(\tau_2 + 0)) + \\ + (S(\tau_2 + 0) - S(\tau_2)) \geq -1 - o(1),$$

so we have $\Delta(n) \geq S(\tau_2) - 1 - o(1)$. Therefore,

$$\Delta(n) - \Delta(m) \geq S(\tau_2) - S(\tau_1) - 2 - o(1) > 4.$$

Thus, the inequality $\Delta(k) - \Delta(k-1) \geq 1$ holds for at least one index k , $k = m+1, \dots, n$. In view of (37), the corresponding Gram's interval G_k contains at least two zeros of $\zeta(0.5 + it)$. This interval lies in the segment $[T_1 - H_1, T_1 + 3H_1 + t_n - t_{n-1}]$ whose length is less than $1.6 \ln \ln \ln T_1 + 4c + 10^{-3}$. Setting $c_0 = 2c + 10^{-3}$, we arrive at the desired assertion. Theorem is proved.

Let $\gamma_n > 0$ be an ordinate of a zero of $\zeta(s)$. Given n , we indicate the unique number $m = m(n)$ such that $t_{m-1} < \gamma_n \leq t_m$. Following Selberg [18], we denote $\Delta_n = m - n$. It is known (see [32, p. 355, remark 1] and [33]) that $\Delta_n \neq 0$ for "almost all" n . Moreover, one can show that the number of indices $n \leq N$ satisfying the condition

$$\Delta_n \leq \frac{x}{\pi\sqrt{2}} \sqrt{\ln \ln N}$$

is expressed as

$$N \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du + O\left(\frac{\ln \ln \ln N}{\sqrt{\ln \ln N}}\right) \right)$$

for any x (see [34, Th. 5] and [35, Th. 4-6]). Given $N \geq N_0$, the above Theorem 3 allows to point out $M = M(N)$ such that the interval $(N, N + M]$ certainly contains an index n with the condition $\Delta_n \neq 0$. Moreover, the following assertion holds.

THEOREM 7. *Suppose that the Riemann hypothesis is true. Then there exist constants N_0 and $c_0 = c_0(\varepsilon)$ such that the interval $(N, N + M]$, where $N \geq N_0$ and*

$$M = \left\lceil \frac{31}{5\pi} (\ln N + c_0) \ln \ln \ln N \right\rceil,$$

contains indices n, m with the conditions $\Delta_n = 3, \Delta_m = -3$.

PROOF. We precede the proof by some remarks.

Firstly, the analogue of intermediate value theorem holds true for the function $S(t)$. Namely, if $\tau_1 < \tau_2$ and $S(\tau_1) > S(\tau_2)$ then for any α with the condition $S(\tau_2) < \alpha < S(\tau_1)$, there exists a point τ on the interval (τ_1, τ_2) such that $S(t)$ is continuous at this point and $S(\tau) = \alpha$ (see [36, proof of Th. 3]).

Secondly, the value $S(t)$ is integer if and only if t is Gram point (see [36, proof of Th. 1]).

Suppose now that T is sufficiently large. By Corollary of Theorem 3, for sufficiently large $c_1 > 0$ and $h = 0.4 \ln \ln \ln T + c_1$, the interval $(T, T + 3h)$ contains the points $\tau_1 < \tau_2$

such that $S(\tau_1) > 3 + 10^{-3}$, $S(\tau_2) < -3 - 10^{-3}$. By the first remark, there exist a point t between τ_1 and τ_2 such that $S(t) = S(t+0) = -3$. By second remark, this point is Gram point, that is, $t = t_{\nu_0}$, $S(t_{\nu_0} + 0) = \Delta(\nu_0) = -3$ for some ν_0 .

Similarly, we prove that each of intervals $(T + (4j - 1)h, T + (4j + 3)h)$, $j = 1, \dots, 5$, contains Gram point t_{ν_j} such that $S(t_{\nu_j} + 0) = \Delta(\nu_j) = -3$. Now we take $T = t_N$. Since

$$h = 0.4 \ln \ln \ln t_N + c_1 < 0.4 \ln \ln \ln N + c_1,$$

then the index ν defined by the relations $t_{N+\nu} < T + 23h \leq t_{N+\nu+1}$, satisfies the following condition:

$$\nu = \frac{1}{\pi} (\vartheta(t_{N+\nu}) - \vartheta(t_N)) < \frac{23h}{\pi} \vartheta'(t_{N+\nu}) < \frac{23h}{2\pi} \ln N < M.$$

Hence, the interval $(N, N + \nu]$ contains at least 6 indices ν_j , $j = 0, \dots, 5$, such that $\Delta(\nu_j) = -3$. It is known (see [35, Lemma 2]) that the number of indices of the same interval satisfying the condition $\Delta_n = 3$ differs from the above quantity for at most $3 + (3 - 1) = 5$ in modulus. Hence, it is positive.

The proof of the second assertion of the Theorem is similar. It uses the fact that the difference between the number of indices n satisfying the condition $\Delta_n = -3$ and the number of indices with the condition $\Delta(\nu) = 3$ lying in the same interval, does not exceed $|-3| + |-3 - 1| = 7$. Theorem is proved.

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